STAT 511 su 2020 Final Exam

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This is a take-home test due to the COVID-19 suspension of face-to-face instruction. Do not communicate with classmates about the exam until after its due date/time. You may

- *Use your notes and the lecture notes.*
- *Use books.*
- *NOT work together with others.*

Write all answers on blank sheets of paper; then take pictures and merge to a PDF. Upload a single PDF to Blackboard.

pmf/pdf	$\mathcal{X}$	$M_X(t)$	$\mathbb{E}X$	Var X
$p_X(x;p) = p^x(1-p)^{1-x}$,	$x = 0, 1$	$pe^t + (1-p)$	p	$p(1-p)$
$p_X(x;n,p) = \binom{n}{x}p^x(1-p)^{n-x}$,	$x = 0, 1, \dots, n$	$[pe^t + (1-p)]^n$	np	$np(1-p)$
$p_X(x;p) = (1-p)^{x-1}p$,	$x = 1, 2, \dots$	$\frac{pe^t}{1-(1-p)e^t}$	p^{-1}	$(1-p)p^{-2}$
$p_X(x;p,r) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$,	$x = r, r+1, \dots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	rp^{-1}	$r(1-p)p^{-2}$
$p_X(x;\lambda) = e^{-\lambda}\lambda^x/x!$	$x = 0, 1, \dots$	$e^{\lambda(e^t-1)}$	λ	λ
$p_X(x;N,M,K) = \binom{M}{x}\binom{N-M}{K-x}/\binom{N}{K}$	$x = 0, 1, \dots, K$	¡complicadísimo!	$\frac{KM}{N}$	$\frac{KM}{N}\frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x;K) = \frac{1}{K}$	$x = 1, \dots, K$	$\frac{1}{K}\sum_{x=1}^{K} e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x;x_1,\dots,x_n) = \frac{1}{n}$	$x = x_1, \dots, x_n$	$\frac{1}{n}\sum_{i=1}^{n} e^{tx_i}$	$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$	$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$
$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2
$f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^{\infty}\frac{t^k}{k!}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

The table below gives some values of the function $\Phi(z) = \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$:

z	0.674	0.841	1.036	1.282	1.645	1.96	2.576
$\Phi(z)$	0.75	0.80	0.85	0.9	0.95	0.975	0.995

1. Copy down this sentence on your answer sheet and put your signature underneath: *I have not collaborated with any other student on this exam. The work I have presented is my own.*

2. Let X and Y each have the Uniform$(0,1)$ distribution, and suppose $\operatorname{corr}(X,Y) = 1/2$.

   (a) Give $\operatorname{Var} X$ and $\operatorname{Var} Y$.

   Solution: The variance of the Uniform$(0,1)$ distribution is $1/12$, so

   $$\operatorname{Var} X = \operatorname{Var} Y = 1/12.$$

   (b) Find $\operatorname{Cov}(X,Y)$.

   Solution: We have

   $$1/2 = \operatorname{corr}(X,Y) = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var} X}\sqrt{\operatorname{Var} Y}} = \frac{\operatorname{Cov}(X,Y)}{1/12},$$

   so that $\operatorname{Cov}(X,Y) = 1/24$.

   (c) Find $\operatorname{Var}(X+Y)$.

   Solution: We have

   $$\operatorname{Var}(X+Y) = \operatorname{Var} X + \operatorname{Var} Y + 2\operatorname{Cov}(X,Y) = 1/12 + 1/12 + 2(1/24) = 3/12 = 1/4.$$

   (d) Find $\operatorname{Var}(X-Y)$.

   Solution: We have

   $$\operatorname{Var}(X-Y) = \operatorname{Var} X + \operatorname{Var} Y - 2\operatorname{Cov}(X,Y) = 1/12 + 1/12 - 2(1/24) = 1/12.$$

   (e) Suppose $\operatorname{corr}(X,Y) = \rho$ for some $\rho \in [-1,1]$. State whether $\mathbb{E}(X-Y)^2$ is an increasing or a decreasing function of ρ. Justify your answer using intuition or by finding the expression for $\mathbb{E}(X-Y)^2$ in terms of ρ.

   Solution: Since $\mathbb{E}X = \mathbb{E}Y$, We have

   $$\mathbb{E}(X-Y)^2 = \operatorname{Var}(X-Y) + [\mathbb{E}(X-Y)]^2 = \operatorname{Var}(X-Y).$$

   In addition, $\operatorname{Cov}(X,Y) = \sqrt{\operatorname{Var} X}\sqrt{VY}\rho = \rho/12$. So we have

   $$\mathbb{E}(X-Y)^2 = \operatorname{Var} X + \operatorname{Var} Y - 2\operatorname{Cov}(X,Y) = 1/12 + 1/12 - 2\rho/12 = (1-\rho)/6.$$

   This is a decreasing function of ρ.

It makes sense that this should be a decreasing function of ρ, because for larger values of ρ we expect more (X, Y) pairs to fall on a line, making $\mathbb{E}(X-Y)^2$, the expected squared distance between X and Y, smaller.

3. Let X and Y be random variables such that

$$Y|X \sim \text{Gamma}(3, 1/X)$$
$$X \sim \text{Gamma}(3, 3).$$

(a) Write down the joint pdf of the random variable pair (X, Y).

Solution: We have

$$\begin{aligned} f(x,y) &= \frac{1}{\Gamma(3)(1/x)^3} y^{3-1} e^{-y/(1/x)} \mathbf{1}(y>0) \cdot \frac{1}{\Gamma(3)3^3} x^{3-1} e^{-x/3} \mathbf{1}(x>0) \\ &= \frac{1}{108} x^5 y^2 e^{-xy-x/3} \mathbf{1}(x>0, y>0). \end{aligned}$$

(b) Write down the integral you would need to solve to obtain the marginal pdf f_Y of Y.

Solution: The marginal pdf f_Y of Y is given by

$$f_Y(y) = \int_0^\infty \frac{1}{108} x^5 y^2 e^{-xy-x/3} dx \mathbf{1}(y>0).$$

(c) Find $\mathbb{E}(1/X)$. *Simplify any gamma functions.*

Solution: We have

$$\begin{aligned} \mathbb{E}(1/X) &= \int_0^\infty \frac{1}{x} \frac{1}{\Gamma(3)3^3} x^{3-1} e^{-x/3} dx \\ &= \frac{\Gamma(2)3^2}{\Gamma(3)3^3} \underbrace{\int_0^\infty \frac{1}{\Gamma(2)3^2} x^{2-1} e^{-x/3} dx}_{=1, \text{ integral over Gamma}(2,3) \text{ pdf}} \\ &= 1/6. \end{aligned}$$

(d) Find $\mathbb{E}Y$. *Hint: Use iterated expectation.*

Solution: Using iterated expectation we have

$$\mathbb{E}Y = \mathbb{E}(\mathbb{E}[Y|X]) = \mathbb{E}(3/X) = 3\mathbb{E}(1/X) = 3/6 = 1/2.$$

(e) Find $\mathbb{E}(1/X^2)$.

Solution: We have

$$\begin{aligned}
\mathbb{E}(1/X^2) &= \int_0^\infty \frac{1}{x^2}\frac{1}{\Gamma(3)3^3}x^{3-1}e^{-x/3}dx \\
&= \frac{\Gamma(1)3^1}{\Gamma(3)3^3}\underbrace{\int_0^\infty \frac{1}{\Gamma(1)3^1}x^{1-1}e^{-x/3}dx}_{=1,\text{ integral over Gamma}(1,3)\text{ pdf}} \\
&= 1/18.
\end{aligned}$$

(f) Find $\operatorname{Var} Y$.

Solution: Using iterated variance, we have

$$\begin{aligned}
\operatorname{Var} Y &= \operatorname{Var}(\mathbb{E}[Y|X]) + \mathbb{E}(\operatorname{Var}[Y|X]) \\
&= \operatorname{Var}(3/X) + \mathbb{E}(3/X^2) \\
&= \mathbb{E}(3/X)^2 - [\mathbb{E}(3/X)]^2 + \mathbb{E}(3/X^2) \\
&= 9/18 - 9/6^2 + 3/18 \\
&= 5/12.
\end{aligned}$$

(g) Find $\operatorname{Cov}(X, Y)$.

Solution: We have

$$\begin{aligned}
\operatorname{Cov}(X,Y) &= \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y \\
&= \mathbb{E}(\mathbb{E}[XY|X]) - \mathbb{E}X\mathbb{E}(\mathbb{E}[Y|X]) \\
&= \mathbb{E}(X\mathbb{E}[Y|X]) - 3\cdot 3\cdot \mathbb{E}(3/X) \\
&= 3 - 27/6 \\
&= -3/2.
\end{aligned}$$

(h) Find $\operatorname{corr}(X, Y)$.

Solution: We have

$$\operatorname{corr}(X,Y) = \frac{-3/2}{\sqrt{\operatorname{Var} X}\sqrt{\operatorname{Var} Y}} = \frac{-3/2}{\sqrt{3\cdot 3^2}\sqrt{15/36}} = -\sqrt{\frac{3}{15}} = -0.4472136.$$

4. Let X_1 and X_2 be independent random variables such that

$$X_1 \sim \text{Gamma}(\alpha_1, \beta)$$
$$X_2 \sim \text{Gamma}(\alpha_2, \beta)$$

for some $\alpha_1, \alpha_2, \beta > 0$.

(a) Find the mgf of the random variable $S = X_1 + X_2$.

Solution: Since X_1 and X_2 are independent, we have

$$M_S(t) = M_{X_1}(t)M_{X_2}(t) = (1 - \beta t)^{-\alpha_1}(1 - \beta t)^{-\alpha_2} = (1 - \beta t)^{-(\alpha_1+\alpha_2)}.$$

(b) Identify the distribution of S.

Solution: The mgf of S is that of the $\text{Gamma}(\alpha_1 + \alpha_2, \beta)$ distribution, so

$$S \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta).$$

(c) Find the mgf of

$$\bar{X} = \frac{X_1 + X_2}{2}.$$

Solution: We have

$$M_{\bar{X}}(t) = M_{S/2}(t) = M_S(t/2) = (1 - \beta(t/2))^{-(\alpha_1+\alpha_2)} = (1 - (\beta/2)t)^{-(\alpha_1+\alpha_2)}.$$

(d) Identify the distribution of $\bar{X}$.

Solution: The mgf of $\bar{X}$ is that of the $\text{Gamma}(\alpha_1 + \alpha_2, \beta/2)$ distribution, so

$$\bar{X} \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta/2).$$

(e) Give $\text{Var}(X_1 - X_2)$.

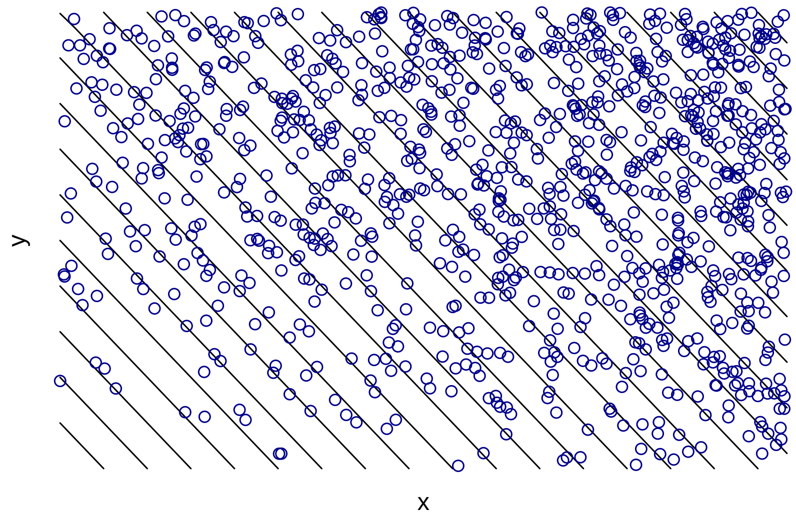
Solution: We have

$$\text{Var}(X_1 - X_2) = \text{Var}\, X_1 + \text{Var}\, X_2 = \alpha_1\beta^2 + \alpha_2\beta^2 = (\alpha_1 + \alpha_2)\beta^2.$$

5. Let the pair of random variables (X, Y) have the joint pdf given by

$$f(x, y) = \frac{2}{ab(a+b)}(x+y) \cdot \mathbf{1}(0 < x < a, 0 < y < b)$$

for some $a, b > 0$. The plot below shows contours of the joint pdf under some specific values of a and b with 1,000 realizations of (X, Y) plotted on top.



(a) State whether X and Y are independent and explain your answer.

Solution: Since we cannot factor the joint pdf into a product of a function of only x and a function of only y, the random variables X and Y are not independent.

(b) Find the marginal pdf f_X of X.

Solution: For $x \in (0, a)$, we have

$$\begin{aligned} f_X(x) &= \int_0^b \frac{2}{ab(a+b)}(x+y)dy \\ &= \frac{2}{ab(a+b)}\left(xy + \frac{y^2}{2}\right)\Big|_0^b \\ &= \frac{2}{ab(a+b)}\left(xb + \frac{b^2}{2}\right) \\ &= \frac{2x+b}{a(a+b)}. \end{aligned}$$

So for all $x \in \mathbb{R}$, we have

$$f_X(x) = \frac{2x+b}{a(a+b)} \cdot \mathbf{1}(0 < x < a).$$

(c) Find the conditional pdf $f(y|x)$ of $Y|X = x$.

Solution: For any $x \in (0, a)$ and $y \in (0, b)$ the conditional pdf of $Y|X = x$ is given by

$$\begin{aligned} f(y|x) &= \frac{2}{ab(a+b)}(x+y) \cdot \left[\frac{2x+b}{a(a+b)}\right]^{-1} \\ &= \frac{2(x+y)}{b(2x+b)}. \end{aligned}$$

(d) Find $P(Y \leq b/2|X = a/2)$.

Solution: We have

$$\begin{aligned} P(Y \leq b/2|X = a/2) &= \int_0^{b/2} \frac{2[(a/2)+y]}{b[2(a/2)+b]} dy \\ &= \frac{1}{b(a+b)} \int_0^{b/2} (a+2y) dy \\ &= \frac{1}{b(a+b)} (ay + y^2) \Big|_0^{b/2} \\ &= \frac{1}{b(a+b)} (ab/2 + b^2/4) \\ &= \frac{2a+b}{4b(a+b)} \end{aligned}$$