STAT 512 su 2021 Lec 01 slides

Transformations of a random variable

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Let X be a rv with support $\mathcal{X}$ and let $Y = g(X)$ for some $g : \mathcal{X} \to \mathcal{Y}$.

Inverse transformation

For any set $A \subset \mathcal{Y}$, define

$$g^{-1}(A) = \{x \in \mathcal{X} : g(x) \in A\}.$$

For a single point $\{y\} \in \mathcal{Y}$, $g^{-1}(\{y\}) = \{x \in \mathcal{X} : g(x) = y\} =: g^{-1}(y)$

Examples:

- $X =$ kWh used in a month. $Y = 0.1125 \times X$, electricity bill.
- $X =$ diameter of sand dollar. $Y = \pi \cdot (X/2)^2$, surface area of underside.
- $X =$ daily rate of return. $Y = A(1 + X)$, value of investment A after a day.
- $X =$ age. $Y = \mathbf{1}(X \geq 62)$, whether eligible for social security in USA.

Distribution of transformed random variable

To find the probability distribution of Y, we note that for any $A \subset \mathcal{Y}$ we have

$$P(Y \in A) = P(g(X) \in A) = P(X \in g^{-1}(A))$$

Exercise: Let X be the 1st roll minus the 2nd roll of a 6-sided die; let $Y = |X|$.

1. Give the support of X.
2. Tabulate the probability distribution of X.
3. Give the support of Y.
4. Tabulate the probability distribution of Y.
5. Find $P(Y \geq 3)$.

Finding the pmf of a transformed discrete rv

If X is discrete with pmf p_X then $Y = g(X)$ is discrete with pmf

$$p_Y(y) = P(X \in g^{-1}(y)) = \sum_{x \in g^{-1}(y)} p_X(x).$$

Any transformation of a discrete rv will result in a discrete rv.

Strategy

1. Find the support $\mathcal{Y}$ of Y.
2. Identify the inverse function $g^{-1}(y)$ for each $y \in \mathcal{Y}$.

Exercise:

1. Let $X \sim \text{Poisson}(\lambda)$. Find the pmf of $Y = \mathbf{1}(X > 10)$.
2. Let $X \sim \text{Binomial}(n, p)$. Find the pmf of $Y = n - X$.

Finding the pdf of a transformed continuous rv by the *cdf method*

Let X be a continuous rv with pdf f_X. Then the cdf of $Y = g(X)$ is given by

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(g(X) \leq y) \\ &= P(X \in \{x : g(X) \leq y\}) \\ &= \int_{\{x: g(x) \leq y\}} f_X(x) dx \end{aligned}$$

Then $f_Y(y) = \frac{d}{dy} F_Y(y)$, provided this is defined over the support of Y.

Exercises:

1. Let $X \sim f_X(x) = (3/2)x^2 \mathbf{1}(-1 \leq x \leq 1)$. Find pdf of $Y = |X|$.
2. Let $X \sim \text{Uniform}(0, 1)$. Find pdf of $Y = -\log X$.

Theorem (*transformation method*)

Let X be a continuous rv with support $\mathcal{X}$ and let $Y = g(X)$ have support $\mathcal{Y}$.

Suppose

1. *g is monotone on $\mathcal{X}$ and*
2. *$\frac{d}{dy} g^{-1}(y)$ is continuous on $\mathcal{Y}$.*

Then

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{d}{dy} g^{-1}(y) \right| \quad \text{for } y \in \mathcal{Y}$$

Exercises: Prove the above and apply to the following:

1. Let $X \sim f_X(x) = (1/\lambda)e^{-x/\lambda}\mathbf{1}(x > 0)$. Find the pdf of $Y = \sqrt{X}$.
2. Let $X \sim \text{Gamma}(\alpha, \beta)$. Find the pdf of $Y = X^{-1}$.
3. Let $X \sim \text{Exponential}(1)$. Find the pdf of $Y = -\log X$.

Exercise: Let $Z \sim \text{Normal}(0,1)$. Find the pdf of $X = Z^2$.

Theorem (Probability integral transform)

Let X be a rv with continuous cdf F_X. Then $Y = F_X(X) \sim \text{Uniform}(0,1)$.

Exercise: Prove the above (assume F_X is monotone).

Random number generation

Apply the probability integral transform in reverse to generate *any* rvs:

1. Generate a $\text{Uniform}(0,1)$ realization.
2. Set $X = F_X^{-1}(U)$, where $F_X^{-1}(u) = \inf\{x : F_X(x) \geq u\}$ is quantile function.

Exercise: Prove the above (assume F_X is monotone).

Many careers spent getting computers to generate "random" $\text{Uniform}(0,1)$ values

Exercise: Get realization of $X \sim \text{Exponential}(\lambda)$ beginning with $U \sim \text{Unif}(0,1)$.

For shift-and-scale transformations, use mgfs (known from STAT 511).

Theorem (*mgf method*)

For any constants a and b, the mgf of $aX + b$ is

$$M_{aX+b}(t) = e^{bt} M_X(at).$$

Exercises:

1. Prove the above.
2. Let $X \sim \text{Gamma}(\alpha, \beta)$. Find distribution of $Y = X/\beta$.
3. Let $X \sim \text{Normal}(\mu, \sigma)$. Find distribution of $Z = (X - \mu)/\sigma$.