TRANSFORMATIONS OF MULTIPLE RANDOM VARIABLES

We are often interested in the distribution of a function involving two or more random variables.

__Example:__ Let X_1, X_2 be independent Exponential(λ) rvs and let

$$Y = \frac{X_1}{X_1 + X_2}.$$

One way to find the distribution of Y is to directly derive its cdf:

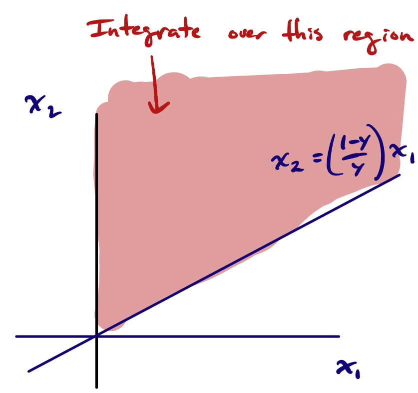
Firstly, the joint density of X_1, X_2 is given by

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\lambda} e^{-x_1/\lambda} \frac{1}{\lambda} e^{-x_2/\lambda} \mathbb{1}(x_1>0)\, \mathbb{1}(x_2>0).$$

Note that the support of Y is $\mathcal{Y} = (0,1)$.

For $y \in (0,1)$, we have

$$\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P\left(\frac{X_1}{X_1+X_2} \le y\right) \\
&= P(X_1 \le y(X_1+X_2)) \\
&= P(X_1(1-y) \le y X_2) \\
&= P\left(X_2 \ge \left(\frac{1-y}{y}\right) X_1\right) \\
&= \int_0^\infty \int_{\left(\frac{1-y}{y}\right)x_1}^\infty \underbrace{\frac{1}{\lambda} e^{-\frac{x_1}{\lambda}} \frac{1}{\lambda} e^{-\frac{x_2}{\lambda}}}_{\text{joint pdf of } X_1, X_2} \, dx_2\, dx_1
\end{aligned}$$



$$= \int_0^\infty \frac{1}{\lambda} e^{-\frac{x_1}{\lambda}} \left[-e^{-x_2/\lambda} \right] \Bigg|_{\left(\frac{1-y}{y}\right)x_1}^{\infty} dx_1$$

$$= \int_0^\infty \frac{1}{\lambda} e^{-x_1/\lambda} e^{-\left(\frac{1-y}{y}\right)x_1/\lambda} dx_1$$

$$= \int_0^\infty \frac{1}{\lambda} e^{-\frac{x_1 y - (1-y)x_1}{y\lambda}} dx_1$$

$$= y \underbrace{\int_0^\infty \frac{1}{y\lambda} e^{-\frac{x_1}{y\lambda}} dx_1}_{=1,\ \text{integral over support of Exponential}(y\lambda)}$$

$$= y.$$

So

$$F_Y(y) = \begin{cases} 0, & y \le 0 \\ y, & 0 < y < 1 \\ 1, & 1 \le y \end{cases}.$$

This is the cdf of the Uniform$(0,1)$ distribution, so

$$Y = \frac{X_1}{X_1 + X_2} \sim \text{Uniform}(0,1).$$

The following theorem is useful in "nice" situations:

Theorem: Let (X_1, X_2) be a pair of continuous rvs with joint pdf f_{X_1,X_2} and support set

$$\mathcal{X} = \{(x_1,x_2) : f_{X_1,X_2}(x_1,x_2) > 0\}.$$

Consider the pair of continuous rvs (Y_1, Y_2) defined as

$$Y_1 = g_1(X_1, X_2), \quad Y_2 = g_2(X_1, X_2),$$

where g_1 and g_2 are one-to-one functions taking values in $\mathcal{X}$ and returning values in the set

$$\mathcal{Y} = \{(y_1,y_2) : y_1 = g_1(x_1,x_2),\ y_2 = g_2(x_1,x_2),\ (x_1,x_2) \in \mathcal{X}\}.$$

Then the joint pdf f_{Y_1,Y_2} of (Y_1, Y_2) is given by

$$f_{Y_1,Y_2}(y_1,y_2) = f_{(X_1,X_2)}\left(g_1^{-1}(y_1,y_2), g_2^{-1}(y_1,y_2)\right) \left| J(y_1,y_2) \right|,$$

( biv)

for $(y_1,y_2) \in \mathcal{Y}$, zero otherwise, where g_1^{-1} and g_2^{-1} are the inverse transformations satisfying

$$\begin{matrix} y_1 = g_1(x_1,x_2) \\ y_2 = g_2(x_1,x_2) \end{matrix} \iff \begin{matrix} x_1 = g_1^{-1}(y_1,y_2) \\ x_2 = g_2^{-1}(y_1,y_2), \end{matrix}$$

and

$$J(y_1,y_2) = \begin{vmatrix} \frac{\partial}{\partial y_1} g_1^{-1}(y_1,y_2) & \frac{\partial}{\partial y_2} g_1^{-1}(y_1,y_2) \\ \frac{\partial}{\partial y_1} g_2^{-1}(y_1,y_2) & \frac{\partial}{\partial y_2} g_2^{-1}(y_1,y_2) \end{vmatrix},$$

provided $J(y_1,y_2)$ is not equal to zero on all of $\mathcal{Y}$.

Remarks

- For real numbers a, b, c, d,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and this called the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- The function $J(y_1, y_2)$ is called the "Jacobian" of the transformation.

- The "nice" part about the situation to which the theorem applies is that the transformations are one-to-one! This theorem could not be used, for example, for the transformation

$$Y_1 = \max\{X_1, X_2\}, \quad Y_2 = \min\{X_1, X_2\},$$

because we cannot get the value of (X_1, X_2) by knowing (Y_1, Y_2).

- The theorem is readily extensible to the n-variable case, where we consider continuous rvs $X_1, \ldots, X_n$ and n transformations $Y_1 = g_1(X_1, \ldots, X_n), \ldots, Y_n = g_n(X_1, \ldots, X_n)$. For this course, however, we will only use the bivariate version.

Example: Let (X_1, X_2) be independent $\text{Normal}(0,1)$ rvs. Find the joint pdf of (Y_1, Y_2), where

$$Y_1 = \frac{X_1}{X_2} \quad \text{and} \quad Y_2 = X_2.$$

The joint pdf of (X_1, X_2) is

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}}.$$

Note that $\mathcal{Y} = (-\infty, \infty) \times (-\infty, \infty)$.

We have

$$\begin{aligned} y_1 &= \frac{x_1}{x_2} \left(= g_1(x_1, x_2)\right) \\ y_2 &= x_2 \left(= g_2(x_1, x_2)\right) \end{aligned} \iff \begin{aligned} x_1 &= y_1 y_2 \left(= g_1^{-1}(y_1, y_2)\right) \\ x_2 &= y_2 \left(= g_2^{-1}(y_1, y_2)\right), \end{aligned}$$

so that

$$J(y_1,y_2) = \begin{vmatrix} \frac{\partial}{\partial y_1} y_1 y_2 & \frac{\partial}{\partial y_2} y_1 y_2 \\ \frac{\partial}{\partial y_1} y_2 & \frac{\partial}{\partial y_2} y_2 \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2 .$$

Applying $(\diamondsuit_{\text{biv}})$ gives

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_1 y_2)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_2^2}{2}} |y_2| .$$

Quite often we are only interested in the marginal pdf of one of the rvs $Y_1 = g_1(X_1,X_2)$ or $Y_2 = g_2(X_1,X_2)$.

Once we have the joint pdf f_{Y_1,Y_2} of (Y_1,Y_2), we may find the marginal pdf of Y_1, say, by taking the integral

$$f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1,Y_2}(y_1,y_2)\, dy_2 .$$

<u>**Example:**</u> Let (X_1,X_2) be independent $\text{Normal}(0,1)$ rvs.
Find the pdf of $Y_1 = \frac{X_1}{X_2}$.

Having found already the joint pdf f_{Y_1,Y_2} of

$$Y_1 = \frac{X_1}{X_2} \quad \text{and} \quad Y_2 = X_2 ,$$

we need only to integrate it over y_2:

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_1 y_2)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_2^2}{2}} |y_2| \, dy_2$$

$$= 2\int_0^{\infty} \frac{1}{2\pi} e^{-\frac{(1+y_1^2)y_2^2}{2}} y_2 \, dy_2 \qquad \left(\int_0^{\infty} e^{-a y_2^2} y_2 \, dy = -\frac{e^{-a y_2^2}}{2a} \Big|_0^{\infty} \right)$$

$$= \frac{1}{\pi}\left[-e^{-\frac{(1+y_1^2)y_2^2}{2}}\left(\frac{1}{1+y_1^2}\right)\right]_0^\infty$$

$$= \frac{1}{\pi}\,\frac{1}{1+y_1^2}\,.$$

This is the pdf of a distribution called the Cauchy distribution.

__Example__: Let $(X_1, X_2) \sim f_{X_1,X_2}(x_1,x_2) = \frac{1}{\lambda^2} e^{-\frac{(x_1+x_2)}{\lambda}} \mathbb{1}(x_1>0,\, x_2>0)$.

(i) Find the joint pdf of $Y_1 = \dfrac{X_1}{X_1+X_2}$ and $Y_2 = X_1+X_2$.

Note that $\mathcal{Y} = \{(y_1,y_2) : 0<y_1<1,\ 0<y_2\}$.

We have

$$\begin{aligned} y_1 &= \frac{x_1}{x_1+x_2} \left(= g_1(x_1,x_2)\right) \\ y_2 &= x_1+x_2 \left(= g_2(x_1,x_2)\right) \end{aligned} \iff \begin{aligned} x_1 &= y_1 y_2 \left(= g_1^{-1}(y_1,y_2)\right) \\ x_2 &= (1-y_1)y_2 \left(= g_2^{-1}(y_1,y_2)\right). \end{aligned}$$

The Jacobian is

$$J(y_1,y_2) = \begin{vmatrix} \frac{\partial}{\partial y_1} y_1y_2 & \frac{\partial}{\partial y_2} y_1y_2 \\ \frac{\partial}{\partial y_1}(1-y_1)y_2 & \frac{\partial}{\partial y_2}(1-y_1)y_2 \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{vmatrix} = y_2(1-y_1) - (-y_2)y_1 = y_2$$

Applying $(\blacklozenge_{\text{biv}})$ gives

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{\lambda^2} e^{-\frac{(y_1y_2+(1-y_1)y_2)}{\lambda}} |y_2|\, \mathbb{1}(0<y_1<1,\ 0<y_2)$$

$$= \frac{y_2}{\lambda^2} e^{-\frac{y_2}{\lambda}} \mathbb{1}(0<y_1<1,\ 0<y_2).$$

(ii) Find the marginal pdf f_{Y_1} of Y_1.

Integrate $f_{Y_1,Y_2}(y_1,y_2)$ over $y_2 \in (0,\infty)$ to get $f_{Y_1}(y_1)$.

$$f_{Y_1}(y_1) = \int_0^\infty \frac{y_2}{\lambda^2} e^{-\frac{y_2}{\lambda}} \, dy_2 \; \mathbb{1}(0<y_1<1)$$

$$= \frac{1}{\lambda} \underbrace{\int_0^\infty \frac{y_2}{\lambda} e^{-\frac{y_2}{\lambda}} \, dy_2}_{=\lambda \text{ Expected value of Exponential}(\lambda) \text{ dist.}} \; \mathbb{1}(0<y_1<1)$$

$$= \mathbb{1}(0<y_1<1)$$

Example: Let $(Z_1,Z_2) \sim f_{Z_1,Z_2}(z_1,z_2) = \frac{1}{2\pi} \frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(z_1^2 - 2\rho z_1 z_2 + z_2^2\right)\right]$.

Find the joint pdf of (U_1,U_2), where

$$U_1 = Z_1 + Z_2 \quad \text{and} \quad U_2 = Z_1 - Z_2 .$$

Firstly (U_1,U_2) has support on $\{(u_1,u_2): u_1 \in \mathbb{R},\, u_2 \in \mathbb{R}\} = \mathbb{R}^2$.

We have a one-to-one transformation:

$$\begin{aligned} u_1 &= z_1 + z_2 \left(= g_1(z_1,z_2)\right) \\ u_2 &= z_1 - z_2 \left(= g_2(z_1,z_2)\right) \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} z_1 &= \frac{u_1+u_2}{2} \left(= g_1^{-1}(u_1,u_2)\right) \\ z_2 &= \frac{u_1-u_2}{2} \left(= g^{-1}(u_1,u_2)\right) \end{aligned}$$

The Jacobian is

$$J(u_1,u_2) = \begin{vmatrix} \frac{\partial}{\partial u_1} \frac{u_1+u_2}{2} & \frac{\partial}{\partial u_2} \frac{u_1+u_2}{2} \\ \frac{\partial}{\partial u_1} \frac{u_1-u_2}{2} & \frac{\partial}{\partial u_2} \frac{u_1-u_2}{2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) - \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = -\tfrac{1}{2} .$$

Applying () gives

$$f_{U_1,U_2}(u_1,u_2) = \frac{1}{2\pi} \frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \left(\left(\frac{u_1+u_2}{2}\right)^2 - 2\rho\left(\frac{u_1+u_2}{2}\right)\left(\frac{u_1-u_2}{2}\right) + \left(\frac{u_1-u_2}{2}\right)^2\right)\right] \left|-\tfrac{1}{2}\right|$$

$$= \frac{1}{2\pi} \frac{1}{\sqrt{4(1-\rho^2)}} \exp\left[-\frac{u_1^2 + 2u_1u_2 + u_2^2 - 2\rho\left(u_1^2 - u_2^2\right) + u_1^2 - 2u_1u_2 + u_2^2}{2 \cdot 4(1-\rho^2)}\right]$$

$4-4\rho^2=(2-2\rho)(2+2\rho)$

$$= \frac{1}{2\pi}\frac{1}{\sqrt{(2-2\rho)(2+2\rho)}}\exp\left[-\frac{u_1^2(2-2\rho)}{2(2-2\rho)(2+2\rho)}\right]\exp\left[-\frac{u_2^2(2+2\rho)}{2(2-2\rho)(2+2\rho)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2+2\rho}}\exp\left[-\frac{u_1^2}{2(2+2\rho)}\right]\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2-2\rho}}\exp\left[-\frac{u_2^2}{2(2-2\rho)}\right].$$

Note that U_1 and U_2 are independent; the joint pdf $f_{U_1,U_2}(u_1,u_2)$ can be factored into the product of a function of only u_1 and a function of only u_2.

Moreover, we see that the marginal distributions are

$$U_1 \sim \text{Normal}(0,\ 2+2\rho)$$
$$U_2 \sim \text{Normal}(0,\ 2-2\rho).$$

Also, from what we knew before we could have found

$$\text{Var}\, U_1 = \text{Var}(Z_1+Z_2) = \text{Var}\, Z_1 + \text{Var}\, Z_2 + 2\,\text{Cov}(Z_1,Z_2) = 2+2\rho$$
$$\text{Var}\, U_2 = \text{Var}(Z_1-Z_2) = \text{Var}\, Z_1 + \text{Var}\, Z_2 - 2\,\text{Cov}(Z_1,Z_2) = 2-2\rho,$$

which confirms our work.

<u>Example</u>: Let $X_1 \sim \text{Beta}(1,1)$ and $X_2 \sim \text{Beta}(2,1)$ be independent rvs.

(i) Find the joint pdf of $Y_1 = X_1X_2$ and $Y_2 = X_2$.

Firstly, the joint pdf of X_1, X_2 is

$$f_{X_1,X_2}(x_1,x_2) = \mathbb{1}(0<x_1<1)\cdot 2x_2\cdot \mathbb{1}(0<x_2<1).$$

We have $\mathcal{X} = (0,1)\times(0,1)$ and $\mathcal{Y} = \{(y_1,y_2): 0<y_1\le y_2<1\}$, and

$$\begin{aligned} y_1 &= x_1x_2 \ (= g_1(x_1,x_2)) \\ y_2 &= x_2 \ (= g_2(x_1,x_2)) \end{aligned} \iff \begin{aligned} x_1 &= y_1/y_2 \ (= g_1^{-1}(y_1,y_2)) \\ x_2 &= y_2 \ (= g_2^{-1}(y_1,y_2)). \end{aligned}$$

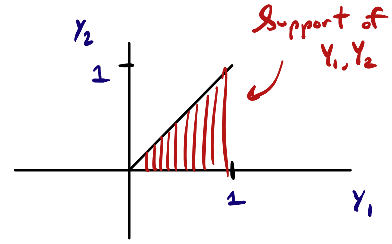
So the Jacobian is

$$J(y_1,y_2) = \begin{vmatrix} \frac{d}{dy_1}\frac{y_1}{y_2} & \frac{d}{dy_2}\frac{y_1}{y_2} \\ \frac{d}{dy_1} y_2 & \frac{d}{dy_2} y_2 \end{vmatrix} = \begin{vmatrix} \frac{1}{y_2} & -\frac{y_1}{y_2^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{y_2}.$$

Then we have, by $(\diamond_{\text{biv}})$, that the joint density of Y_1, Y_2 is

$$f_{Y_1,Y_2}(y_1,y_2) = 2\cdot y_2 \left|\frac{1}{y_2}\right| \cdot \mathbb{1}(0 < y_1 \le y_2 < 1)$$

$$= 2\,\mathbb{1}(0 < y_1 \le y_2 < 1).$$



(ii) Find the marginal density of Y_1.

$$f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1,Y_2}(y_1,y_2)\,dy_2 = \int_{\mathbb{R}} 2\,\mathbb{1}(0<y_1 \le y_2 < 1)\,dy_2 = 2(1-y_1)\,\mathbb{1}(0<y_1<1).$$

If a transformation is not one-to-one, we cannot use expression $(\diamond_{\text{biv}})$. The following is an example of such a situation:

<u>Example</u>: Let $(Z_1,Z_2) \sim f_{Z_1,Z_2}(z_1,z_2) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(z_1^2 - 2\rho z_1 z_2 + z_2^2\right)\right]$

Consider the transformation

$$U_1 = \min\{Z_1, Z_2\} \quad \text{and} \quad U_2 = \max\{Z_1,Z_2\}.$$

Firstly, (U_1,U_2) has support on $\mathbb{R}^2$.

We find that $(u_1,u_2) = (\min\{z_1,z_2\}, \max\{z_1,z_2\}) \left(= g(z_1,z_2)\right)$ has the inverse mapping

$$g^{-1}(u_1,u_2) = \{(z_1,z_2) : (z_1,z_2) \in \{(u_1,u_2),(u_2,u_1)\}\}$$

which is not single-valued, so the transformation is not one-to-one, and we cannot use $(\diamond_{\text{biv}})$.

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Finding the joint pdf of (U_1, U_2) is quite complicated, but we may find the marginal pdf of U_2 say, as follows, beginning by writing down its cdf:

$$
\begin{aligned}
F_{U_2}(u_2) &= P_{U_2}(U_2 \le u_2) \\
&= P_{(Z_1,Z_2)}\left(\max\{Z_1, Z_2\} \le u_2\right) \\
&= P_{(Z_1,Z_2)}\left(Z_1 \le u_2,\ Z_2 \le u_2\right) \\
&= \int_{-\infty}^{u_2}\int_{-\infty}^{u_2} \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\underbrace{\left(z_1^2 - 2\rho z_1 z_2 + z_2^2\right)}_{=(z_1-\rho z_2)^2 + z_2^2(1-\rho^2)}\right] dz_1\, dz_2 \\
&= \int_{-\infty}^{u_2}\int_{-\infty}^{u_2} \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{(z_1-\rho z_2)^2}{2(1-\rho^2)}\right] dz_1\, \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z_2^2}{2}\right] dz_2
\end{aligned}
$$

$\left(\text{Let } \phi(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \text{ and } \Phi(v) = \int_{-\infty}^{v} \phi(v)\, dv\right)$

$$
= \int_{-\infty}^{u_2} \Phi\left(\frac{u_2 - \rho z_2}{\sqrt{1-\rho^2}}\right) \phi(z_2)\, dz .
$$

Now we may find the pdf $f_{U_2}(u_2)$ of U_2 by taking the derivative of $F_{U_2}(u_2)$, but we need to use Leibniz's rule, as u_2 appears in the integrand as well as in the integration bounds.

Leibniz's rule:

$$
\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\, dt\right) = f(x, b(x))\frac{d}{dx}b(x) - f(x, a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)\, dt
$$

This gives

$$
\frac{d}{du_2}\left(\int_{-\infty}^{u_2} \Phi\left(\frac{u_2-\rho z_2}{\sqrt{1-\rho^2}}\right)\phi(z_2)\, dz_2\right) = \Phi\left(\frac{u_2-\rho u_2}{\sqrt{1-\rho^2}}\right)\phi(u_2)\cdot 1 - [0]
$$

$$
+ \int_{-\infty}^{u_2} \frac{1}{\sqrt{1-\rho^2}}\, \phi\left(\frac{u_2-\rho z_2}{\sqrt{1-\rho^2}}\right)\phi(z_2)\, dz_2 ,
$$

where

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$$\int_{-\infty}^{u_2} \frac{1}{\sqrt{1-\rho^2}} \phi\left(\frac{u_2-\rho z_2}{\sqrt{1-\rho^2}}\right) \phi(z_2)\, dz_2 = \int_{-\infty}^{u_2} \frac{1}{\sqrt{1-\rho^2}} \frac{1}{2\pi} \exp\left[-\frac{1}{2} \frac{(u_2-\rho z_2)^2 + z_2^2(1-\rho^2)}{1-\rho^2}\right] dz_2$$

$$= \int_{-\infty}^{u_2} \frac{1}{\sqrt{1-\rho^2}} \frac{1}{2\pi} \exp\left[-\frac{1}{2} \frac{u_2^2 - 2\rho u_2 z_2 + z_2^2}{1-\rho^2}\right] dz_2$$

$$= \int_{-\infty}^{u_2} \frac{1}{\sqrt{1-\rho^2}} \frac{1}{2\pi} \exp\left[-\frac{1}{2} \frac{(z_2-\rho u_2)^2 + u_2^2(1-\rho^2)}{1-\rho^2}\right] dz_2$$

$$= \int_{-\infty}^{u_2} \frac{1}{\sqrt{1-\rho^2}} \phi\left(\frac{z_2-\rho u_2}{\sqrt{1-\rho^2}}\right) \phi(u_2)\, dz_2$$

$$= \phi(u_2)\, \Phi\left(u_2 \frac{1-\rho}{\sqrt{1-\rho^2}}\right).$$

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So in the end, the pdf of the maximum of Z_1 and Z_2, when (Z_1, Z_2) is a standard bivariate Normal pair with correlation ρ, is given by

$$f_{U_2}(u_2) = 2\, \Phi\left(u_2 \sqrt{\frac{1-\rho}{1+\rho}}\right) \phi(u_2),$$

noting that $(1-\rho)/\sqrt{1-\rho^2} = \sqrt{(1-\rho)/(1+\rho)}$.

SUMS OF INDEPENDENT RANDOM VARIABLES

We are very often interested in the distribution of the sum of several independent random variables.

The following result about the mgf of the sum of several independent random variables provides an easy way to get such a distribution.

Theorem: Let $X_1, \dots, X_n$ be mutually independent rvs such that X_i has mgf $M_{X_i}(t)$ for $i = 1, \dots, n$. Let $Y_n = X_1 + \dots + X_n$. Then the mgf of Y_n is

← We often suppress the word "mutual"

$$M_{Y_n}(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

In particular, if $X_1, \dots, X_n$ all have the same mgf $M_X(t)$, then

$$M_{Y_n}(t) = \left[M_X(t)\right]^n.$$

Proof: We have

$$\begin{aligned} M_{Y_n}(t) &= \mathbb{E}\, e^{tY_n} \\ &= \mathbb{E}\, e^{t(X_1 + \dots + X_n)} \\ &= \mathbb{E}\left[e^{tX_1} \cdot \dots \cdot e^{tX_n}\right] \\ &= \mathbb{E}\, e^{tX_1} \cdot \dots \cdot \mathbb{E}\, e^{tX_n} \\ &= M_{X_1}(t) \cdot \dots \cdot M_{X_n}(t). \end{aligned}$$

By independence

Example: Let $X_1, \dots, X_n$ be independent r.v.s such that $X_i \sim \text{Chi-square}(\nu_i)$ for $i = 1, \dots, n$.

Find the distribution of $Y_n = X_1 + \dots + X_n$.

The Chi-square(ν_i) distribution is the same as the Gamma$(\frac{\nu_i}{2}, 2)$ distribution, so X_i has mgf

$$M_{X_i}(t) = (1-2t)^{-\nu_i/2}, \quad i = 1, \dots, n.$$

Therefore

$$\begin{aligned} M_{Y_n}(t) &= \prod_{i=1}^{n} (1-2t)^{-\nu_i/2} \\ &= (1-2t)^{-\sum_{i=1}^{n} \nu_i/2}, \end{aligned}$$

which is the mgf of the Gamma$\left(\sum_{i=1}^{n} \nu_i / 2, 2\right)$ distribution, i.e. the Chi-square$\left(\sum_{i=1}^{n} \nu_i\right)$ distribution.

We may remember from this example that the sum of independent Chi-square variables is a Chi-square, where the degrees of freedom are cumulative.

Example: Let $X_1, \ldots, X_n$ be independent rvs such that $X_i \sim \text{Normal}(\mu_i, \sigma_i^2)$ for $i = 1, \ldots, n$.

(i) Find the distribution of $Y_n = X_1 + \cdots + X_n$.

We have

$$M_{Y_n}(t) = \prod_{i=1}^{n} e^{\mu_i t + \sigma_i^2 t^2/2}$$

$$= e^{\left(\sum_{i=1}^{n} \mu_i\right) t + \left(\sum_{i=1}^{n} \sigma_i^2\right) t^2/2},$$

which is the mgf of the $\text{Normal}\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$ distribution.

We can remember from this example that the sum of independent Normals is Normal, where means and variances have "accumulated."

(ii) Find the distribution of $Y_n' = a_1 X_1 + \cdots + a_n X_n$, where $a_1, \ldots, a_n$ are real numbers.

$$M_{Y_n'}(t) = \prod_{i=1}^{n} M_{a_i X_i}(t)$$

$$= \prod_{i=1}^{n} M_{X_i}(t a_i)$$

$$= \prod_{i=1}^{n} e^{\mu_i (t a_i) + \sigma_i^2 (t a_i)^2/2}$$

$$= e^{\left(\sum_{i=1}^{n} a_i \mu_i\right) t + \left(\sum_{i=1}^{n} a_i^2 \sigma_i^2\right) t^2/2},$$

which is the mgf of the $\text{Normal}\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$ distribution.

Example: Let $X_1, \ldots, X_n$ be independent rvs such that $X_i \sim \text{Normal}(\mu, \sigma^2)$ for $i = 1, \ldots, n$.

Find the distribution of $\bar{X}_n = \frac{1}{n}(X_1 + \ldots + X_n)$.

$$M_{\bar{X}_n}(t) = \prod_{i=1}^{n} M_{\frac{1}{n}X_i}(t)$$

$$= \prod_{i=1}^{n} M_{X_i}(t/n)$$

$$= \prod_{i=1}^{n} e^{\mu(t/n) + \sigma^2 (t/n)^2/2}$$

$$= e^{\left(\sum_{i=1}^{n} \frac{\mu}{n}\right)t + \left(\sum_{i=1}^{n} \frac{\sigma^2}{n^2}\right)\frac{t^2}{2}}$$

$$= e^{\mu t + \left(\frac{\sigma^2}{n}\right)t^2/2},$$

which is the mgf of the $\text{Normal}\left(\mu, \frac{\sigma^2}{n}\right)$ distribution.