STAT 512 su 2021 Lec 02 slides

Transformations of multiple random variables

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

We often wish to find the distribution of a function of two (or more) rvs:

Exercise: Let X_1 and X_2 be indep. Exponential(λ) rvs. Let $Y = X_1/(X_1 + X_2)$.

1. Find the cdf of Y.
2. Find the pdf of Y.

Theorem (*Bivariate transformation method*)

Let (X_1, X_2) *be a pair of cont. rvs with joint pdf* f_{X_1,X_2} *on* $\mathcal{X}$ *and*

$$Y_1 = g_1(X_1, X_2) \quad \textit{and} \quad Y_2 = g_2(X_1, X_2),$$

where g_1 *and* g_2 *define a 1:1 transformation of* $\mathcal{X}$ *onto* $\mathcal{Y}$ *(define these).*

Let g_1^{-1} *and* g_2^{-1} *be the functions satisfying*

$$\begin{array}{l} y_1 = g_1(x_1, x_2) \\ y_2 = g_2(x_1, x_2) \end{array} \iff \begin{array}{l} x_1 = g_1^{-1}(y_1, y_2) \\ x_2 = g_2^{-1}(y_1, y_2) \end{array}$$

for all $(x_1, x_2) \in \mathcal{X}$ *and* $(y_1, y_2) \in \mathcal{Y}$.

Then the joint pdf of (Y_1, Y_2) *is given by*

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2))|J(y_1, y_2)|, \ \textit{for } (y_1, y_2) \in \mathcal{Y},$$

where $J(y_1, y_2)$ *is the* Jacobian *(next slide), if* $J(y_1, y_2)$ *is not always* 0 *on* $\mathcal{Y}$.

Jacobian

In the setup of the previous slide, the *Jacobian* of the transformation is defined as

$$J(y_1, y_2) = \begin{vmatrix} \frac{\partial}{\partial y_1} g_1^{-1}(y_1, y_2) & \frac{\partial}{\partial y_2} g_1^{-1}(y_1, y_2) \\ \frac{\partial}{\partial y_1} g_2^{-1}(y_1, y_2) & \frac{\partial}{\partial y_2} g_2^{-1}(y_1, y_2) \end{vmatrix}.$$

For real numbers a, b, c, d,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

This is called the *determinant*.

Exercise: Let X_1, X_2 be independent $\text{Normal}(0, 1)$ rvs.

1. Find the joint pdf of $Y_1 = X_1/X_2$ and $Y_2 = X_2$.
2. Find the marginal pdf of Y_1.

Exercise: Let X_1, X_2 have joint pdf

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\lambda^2}\exp\left[-\frac{x_1+x_2}{\lambda}\right]\mathbf{1}(x_1>0, x_2>0).$$

1. Find the joint pdf of $Y_1 = X_1/(X_1+X_2)$ and $Y_2 = X_1 + X_2$.
2. Find the marginal pdf of Y_1.

Exercise: Let $X_1 \sim \text{Beta}(1,1)$ and $X_2 \sim \text{Beta}(2,1)$ be independent rvs.

1. Find the joint pdf of $Y_1 = X_1 X_2$ and $Y_2 = X_2$.
2. Find the marginal pdf of Y_1.

Exercise: Let Z_1, Z_2 have the bivariate Normal distribution, with joint pdf

$$f_{Z_1,Z_2}(z_1, z_2) = \frac{1}{2\pi} \frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2} \frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{1-\rho^2}\right].$$

1. Find the joint pdf of $U_1 = Z_1 + Z_2$ and $U_2 = Z_1 - Z_2$.
2. Find the marginal pdfs of U_1 and U_2.

Exercise: Let Z_1, Z_2 have the bivariate Normal distribution, with joint pdf

$$f_{Z_1,Z_2}(z_1, z_2) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2}\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{1-\rho^2}\right].$$

1. Find the joint pdf of $U_1 = \min\{Z_1, Z_2\}$ and $U_2 = \max\{Z_1, Z_2\}$? (Not 1:1).
2. Find the marginal pdf of U_2 (can skip this example; see notes if curious).

Theorem (mgf method for sums of independent rvs)

Let $X_1, \dots, X_n$ *be ind. rvs with mgfs* $M_{X_1}, \dots, M_{X_n}$*, resp. Let* $Y = X_1 + \cdots + X_n$.

The mgf of Y *is given by*

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

Moreover, if $X_1, \dots, X_n$ *are ind. and all have mgf* M_X *(are iid), then*

$$M_Y(t) = [M_X(t)]^n.$$

Exercise: Prove the above.

Exercise: Let $X_1, \ldots, X_n$ be ind. chi-squared rvs with dfs $\nu_1, \ldots, \nu_n$, resp. Find the distribution of $Y = X_1 + \cdots + X_n$.

Exercise: Let $X_1, \ldots, X_n$ be ind. Normal rvs with means $\mu_1, \ldots, \mu_n$ and variances $\sigma_1^2, \ldots, \sigma_n^2$, resp.

1. Find the distribution of $Y = X_1 + \cdots + X_n$.
2. Find the distribution of $V = a_1 X_1 + \cdots + a_n X_n$, for $a_1, \ldots, a_n \in \mathbb{R}$.
3. Find the distribution of $\bar{X}_n = (X_1 + \cdots + X_n)/n$.