PIVOT QUANTITIES AND SAMPLING FROM THE NORMAL DISTRIBUTION

We often study the sampling distributions of so-called pivot quantities.

Defn: A pivot quantity is a function of sample statistics and population parameters which has a known distribution.

Pivot quantities are useful for constructing confidence intervals for unknown parameters and for testing hypotheses about them.

Defn: A $(1-\alpha)*100\%$ confidence interval (C.I.) for an unknown parameter θ is an interval (L, U) where L and U are random variables such that $P(L < \theta < U) = 1-\alpha$, for $\alpha \in (0,1)$.

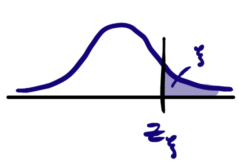
Example: If $X_1, \ldots, X_n$ is a random sample from the $\text{Normal}(\mu, \sigma^2)$ distribution, then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \underbrace{\text{Normal}(0,1)}_{\text{This is fully known, not having any unknown parameters.}}.$$

So $\dfrac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$ is a pivot quantity.

Introducing $Z \sim \text{Normal}(0,1)$, we can construct a $(1-\alpha)*100\%$ C.I. for μ as follows:

Let z_ξ be the value which satisfies $P_Z(Z > z_\xi) = \xi$ for $\xi \in (0,1)$.
Then



$$P_{\bar{X}_n}\left(z_{1-\alpha/2} < \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2} \right) = 1-\alpha$$

$$\Longleftrightarrow \quad P_{\bar{X}_n}\left(\bar{X}_n + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \right) = 1-\alpha,$$

Giving, since $z_{1-\alpha/2} = -z_{\alpha/2}$, the $(1-\alpha)*100\%$ C.I.

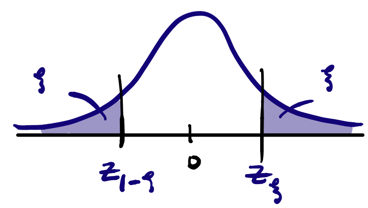
$$\left(\bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \right).$$

NOW WE INTRODUCE FOUR RELEVANT DISTRIBUTIONS:

(These are distributions of some important pivot quantities)

1. The Standard Normal distribution: Normal(0,1)

pdf: $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$



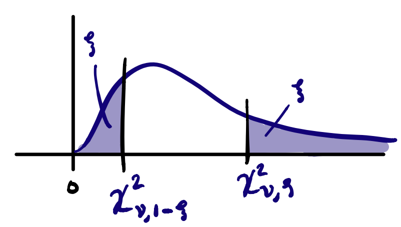
mgf: $M_Z(t) = e^{t^2/2}$

Let z_ξ be the value satisfying $P_Z(Z > z_\xi) = \xi$ for $\xi \in (0,1)$, where $Z \sim \text{Normal}(0,1)$.

2. The Chi-square distributions: χ^2_ν, $\nu = 1, 2, \ldots$

"degrees of freedom"

pdf: $f_X(x;\nu) = \frac{1}{\Gamma(\nu/2)\, 2^{\nu/2}} x^{\nu/2 - 1} e^{-x/2} \mathbb{1}(x>0)$



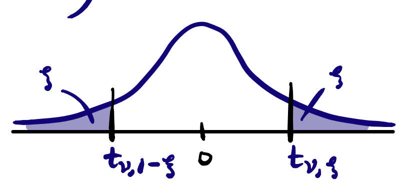
mgf: $(1-2t)^{-\nu/2}$, $t < 1/2$

Let $\chi^2_{\nu,\xi}$ be the value satisfying $P_X(X > \chi^2_{\nu,\xi}) = \xi$ for $\xi \in (0,1)$, where $X \sim \chi^2_\nu$.

3. The t distributions: t_ν, $\nu = 1, 2, \ldots$

"degrees of freedom"

pdf: $f_T(t;\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$



mgf: does not exist!
(moments of order ν or higher do not exist)

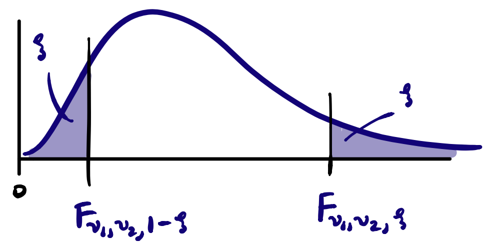
Let $t_{\nu,\xi}$ be the value satisfying $P_T(T > t_{\nu,\xi}) = \xi$ for $\xi \in (0,1)$, where $T \sim t_\nu$.

"numerator degrees of freedom" "denominator degrees of freedom"

4. <u>The F distributions:</u> F_{ν_1,ν_2}, $\nu_1 = 1, 2, \ldots$, $\nu_2 = 1, 2, \ldots$

pdf: $$f_R(r;\nu_1,\nu_2) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}\frac{r^{\frac{\nu_1-2}{2}}}{\left(1+\left(\frac{\nu_1}{\nu_2}\right)r\right)^{\frac{\nu_1+\nu_2}{2}}}\,\mathbb{1}(r>0)$$

mgf: does not exist!



Let $F_{\nu_1,\nu_2,\xi}$ be the value satisfying $P_R(R > F_{\nu_1,\nu_2,\xi}) = \xi$ for $\xi \in (0,1)$, where $R \sim F_{\nu_1,\nu_2}$.

We now present four important pivot-quantity sampling distribution results in a theorem.

<u>Theorem:</u> Let $X_1, \ldots, X_n$ be a random sample from the Normal(μ, σ^2) distribution and let

$$\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$$
$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

Then

I. $$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0,1)$$

Application to C.I.s:

$$P_{\bar{X}_n}\left(-z_{\alpha/2} < \frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = 1-\alpha$$

$\Leftrightarrow$

$$P_{\bar{X}_n}\left(\bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha,$$

$\Leftrightarrow$

A $(1-\alpha)*100\%$ C.I. for μ is $\bar{X}_n \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

II. $\bar{X}_n$ and S_n^2 are independent and $\dfrac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$

Application to C.I.s:

$$P_{S_n^2}\left(\chi^2_{n-1,1-\alpha/2} < \frac{(n-1)S_n^2}{\sigma^2} < \chi^2_{n-1,\alpha/2}\right) = 1-\alpha$$

$\Leftrightarrow$

$$P_{S_n^2}\left(\frac{(n-1)S_n^2}{\chi^2_{n-1,\alpha/2}} < \sigma^2 < \frac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha/2}}\right) = 1-\alpha$$

$\Leftrightarrow$

A $(1-\alpha)^*100\%$ C.I. for σ^2 is $\left(\dfrac{(n-1)S_n^2}{\chi^2_{n-1,\alpha/2}}, \dfrac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha/2}}\right)$

III. $$\frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{n-1}$$

Application to C.I.s:

$$P_{\bar{X}_n,S_n}\left(-t_{n-1,\alpha/2} < \frac{\bar{X}_n-\mu}{S_n/\sqrt{n}} < t_{n-1,\alpha/2}\right) = 1-\alpha$$

$\Leftrightarrow$

$$P_{\bar{X}_n,S_n}\left(\bar{X}_n - t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}} < \mu < \bar{X}_n + t_{n-1,\alpha/2}\frac{S_n}{\sqrt{n}}\right) = 1-\alpha$$

$\Leftrightarrow$

A $(1-\alpha)^*100\%$ C.I. for μ is $\bar{X}_n \pm t_{n-1,\alpha/2}\dfrac{S_n}{\sqrt{n}}$.

IV. Suppose we have two independent random samples

$$X_1,\dots,X_{n_1} \overset{ind}{\sim} \text{Normal}(\mu_1, \sigma_1^2)$$
$$Y_1,\dots,Y_{n_2} \overset{ind}{\sim} \text{Normal}(\mu_2, \sigma_2^2),$$

and let

$$S_1^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}\left(X_i - \bar{X}_{n_1}\right)^2 \quad \text{and} \quad S_2^2 = \frac{1}{n_2-1}\sum_{i=1}^{n_2}\left(Y_i - \bar{Y}_{n_2}\right)^2.$$

Then $$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1,n_2-1}.$$

Application to C.I.s:

$$P_{S_1^2,S_2^2}\left(F_{n_1-1,n_2-1,1-\alpha/2} < \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < F_{n_1-1,n_2-1,\alpha/2}\right) = 1-\alpha$$

$\Leftrightarrow$

$$P_{S_1^2,S_2^2}\left(\frac{S_2^2}{S_1^2}F_{n_1-1,n_2-1,1-\alpha/2} < \frac{\sigma_2^2}{\sigma_1^2} < \frac{S_2^2}{S_1^2}F_{n_1-1,n_2-1,\alpha/2}\right) = 1-\alpha$$

$\Leftrightarrow$

A $(1-\alpha)^*100\%$ C.I. for $\frac{\sigma_2^2}{\sigma_1^2}$ is

$$\left(\frac{S_2^2}{S_1^2}F_{n_1-1,n_2-1,1-\alpha/2}\ ,\ \frac{S_2^2}{S_1^2}F_{n_1-1,n_2-1,\alpha/2}\right).$$

Proofs of I and II of theorem:

I. Proof of $\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0,1)$:

First find the mgf of $\bar{X}_n$:

$$M_{\bar{X}_n}(t) = M_{\frac{1}{n}(X_1+\cdots+X_n)}(t)$$

$$= M_{X_1+\cdots+X_n}(t/n)$$

$$= \left[e^{\mu(t/n) + \sigma^2(t/n)^2/2}\right]^n$$

$$= e^{\mu t + \frac{\sigma^2}{n}t^2/2}$$

Now

$$M_{\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}}}(t) = M_{\frac{\sqrt{n}}{\sigma}\bar{X} - \frac{\sqrt{n}}{\sigma}\mu}(t)$$

$$= e^{-\frac{\sqrt{n}}{\sigma}\mu t}\, M_{\bar{X}_n}\left(\frac{\sqrt{n}}{\sigma}t\right)$$

$$= e^{-\frac{\sqrt{n}}{\sigma}\mu t}\, e^{\mu\frac{\sqrt{n}}{\sigma}t + \frac{\sigma^2}{n}\left(\frac{\sqrt{n}}{\sigma}t\right)^2/2}$$

$$= e^{t^2/2},$$

which is the mgf of the Normal(0,1) distribution. $\square$

II. Proof of $\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$.

(We will not prove that $\bar{X}_n$ and S_n^2 are independent, as the proof is too advanced for this course. Note that $\bar{X}$ and S^2 are independent if and only if the random sample comes from a Normal distribution.)

Begin by re-expressing $\frac{(n-1)S_n^2}{\sigma^2}$:

$$\begin{aligned}
\frac{(n-1)S_n^2}{\sigma^2} &= \frac{(n-1)}{\sigma^2}\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2 \\
&= \sum_{i=1}^n \left(\frac{X_i - \bar{X}_n}{\sigma}\right)^2 \\
&= \sum_{i=1}^n \left(\frac{X_i - \mu + \mu - \bar{X}_n}{\sigma}\right)^2 \\
&= \sum_{i=1}^n \left(\frac{X_i-\mu}{\sigma}\right)^2 + \underbrace{2\sum_{i=1}^n \left(\frac{X_i-\mu}{\sigma}\right)\left(\frac{\mu-\bar{X}_n}{\sigma}\right)}_{= -2\left(\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}}\right)^2} + \underbrace{\sum_{i=1}^n \left(\frac{\mu-\bar{X}_n}{\sigma}\right)^2}_{=\left(\frac{\mu-\bar{X}_n}{\sigma/\sqrt{n}}\right)^2} \\
&= \sum_{i=1}^n \left(\frac{X_i-\mu}{\sigma}\right)^2 - \left(\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}}\right)^2,
\end{aligned}$$

so that

$$\underbrace{\left(\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}}\right)^2}_{\chi^2_1} + \frac{(n-1)S_n^2}{\sigma^2} = \underbrace{\sum_{i=1}^n \left(\frac{X_i-\mu}{\sigma}\right)^2}_{\text{Sum of } n \text{ indep. } \chi^2_1\text{, so this has the } \chi^2_n \text{ dist.}}$$

Recall that $Z^2 \sim \chi^2_1$ if $Z \sim \text{Normal}(0,1)$ (see Lec 01) and that the sum of chi-square rvs is a chi-square rv with the total degrees of freedom (see Lec 02).

Since $\bar{X}_n$ and S_n^2 are independent, the rvs

$$\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)^2 \quad \text{and} \quad \frac{(n-1)S_n^2}{\sigma^2} \quad \text{are independent,}$$

so the mgf of the left hand side is the product of the mgfs of the two rvs.

Therefore

$$(1-2t)^{-1/2} M_{\frac{(n-1)S_n^2}{\sigma^2}}(t) = (1-2t)^{-n/2}$$

$\iff$

$$M_{\frac{(n-1)S_n^2}{\sigma^2}}(t) = (1-2t)^{-\frac{(n-1)}{2}},$$

which is the mgf of the χ^2_{n-1} distribution. $\square$

Before proving parts III and IV of the theorem, we will study the "anatomy" of random variables having a t or an F distribution.

Result: Let $Z \sim \text{Normal}(0,1)$ and $W \sim \chi^2_\nu$, Z, W independent.

Then $T = Z/\sqrt{W/\nu} \sim t_\nu$.

Remark: So a t distribution arises when a Normal$(0,1)$ rv is divided by the square root of an independent chi-square rv divided by its degrees of freedom.

Proof: We first get the joint pdf of (T, U), where

$$T = Z/\sqrt{W/\nu} \quad \text{and} \quad U = W.$$

The joint pdf of Z and W is

$$f_{(Z,W)}(z,w) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \frac{1}{\Gamma(\nu/2)\, 2^{\nu/2}} w^{\nu/2 - 1} e^{-w/2} \mathbb{1}(w > 0).$$

The support of (T,U) is $\{(t,u): -\infty < t < \infty,\ 0 < u < \infty\}$, and

$$\begin{aligned} T &= Z/\sqrt{W/\nu} \quad (= g_1(z,w)) \\ U &= W \quad (= g_2(z,w)) \end{aligned} \iff \begin{aligned} Z &= T\sqrt{U/\nu} \quad (= g_1^{-1}(t,u)) \\ W &= U \quad (= g_2^{-1}(t,u)), \end{aligned}$$

giving the Jacobian

$$J(t,u) = \begin{vmatrix} \frac{\partial}{\partial t} t\sqrt{u/\nu} & \frac{\partial}{\partial u} t\sqrt{u/\nu} \\ \frac{\partial}{\partial t} u & \frac{\partial}{\partial u} u \end{vmatrix} = \begin{vmatrix} \sqrt{u/\nu} & t\left(\frac{1}{2}\right)\frac{1}{\sqrt{u}\sqrt{\nu}} \\ 0 & 1 \end{vmatrix} = \sqrt{u/\nu}\,.$$

Now we get

$$f_{(T,U)}(t,u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{t^2 u}{\nu}} \frac{1}{\Gamma(\nu/2)\, 2^{\nu/2}} u^{\nu/2-1} e^{-u/2} \sqrt{u/\nu}\ \mathbb{1}(u>0)\,.$$

Now we integrate $f_{(T,U)}$ over u to get f_T:

$$\begin{aligned} f_T(t) &= \int_0^\infty \frac{1}{\Gamma(\nu/2)} \frac{1}{\sqrt{\nu\pi}} \frac{1}{2^{\frac{\nu+1}{2}}} u^{\frac{\nu+1}{2}-1} e^{-\frac{u}{2}\left(1+\frac{t^2}{\nu}\right)} du \\ &= \int_0^\infty \frac{1}{\Gamma(\nu/2)\sqrt{\nu\pi}} \frac{\left[\frac{1}{2}\left(1+\frac{t^2}{\nu}\right)\right]^{-\left(\frac{\nu+1}{2}\right)}}{2^{\frac{\nu+1}{2}}} \frac{u^{\frac{\nu+1}{2}-1} e^{-u/\left[\frac{1}{2}\left(1+t^2/\nu\right)\right]^{-1}}}{\left[\frac{1}{2}\left(1+\frac{t^2}{\nu}\right)\right]^{-\left(\frac{\nu+1}{2}\right)}} du \\ &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\nu/2)} \frac{1}{\sqrt{\nu\pi}} \frac{1}{\left(1+t^2/\nu\right)^{\frac{\nu+1}{2}}} \underbrace{\int_0^\infty \frac{u^{\frac{\nu+1}{2}-1} e^{-u/\left[\frac{1}{2}\left(1+t^2/\nu\right)\right]^{-1}}}{\Gamma\left(\frac{\nu+1}{2}\right)\left[\frac{1}{2}\left(1+\frac{t^2}{\nu}\right)\right]^{-\left(\frac{\nu+1}{2}\right)}} du}_{=1,\ \text{integral over pdf of Gamma}\left(\frac{\nu+1}{2},\left[\frac{1}{2}\left(1+\frac{t^2}{\nu}\right)\right]^{-1}\right)} \\ &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\nu/2)} \frac{1}{\sqrt{\nu\pi}} \frac{1}{\left(1+t^2/\nu\right)^{\frac{\nu+1}{2}}}, \end{aligned}$$

which is the pdf of the t_ν distribution. $\square$

Result: Let $W_1 \sim \chi^2_{\nu_1}$ and $W_2 \sim \chi^2_{\nu_2}$, W_1, W_2 independent.

Then $R = \dfrac{W_1/\nu_1}{W_2/\nu_2} \sim F_{\nu_1, \nu_2}$.

Remark: An F distribution arises when the ratio is taken of two independent Chi-square rvs, each divided by its degrees of freedom!

Proof: We first get the joint pdf of (R, U), where

$$R = \frac{W_1/\nu_1}{W_2/\nu_2} \quad \text{and} \quad U = W_2 .$$

The joint pdf of (W_1, W_2) is

$$f_{(W_1,W_2)}(w_1,w_2) = \frac{1}{\Gamma(\frac{\nu_1}{2}) 2^{\frac{\nu_1}{2}}} w_1^{\nu_1/2 - 1} e^{-\frac{w_1}{2}} \frac{1}{\Gamma(\frac{\nu_2}{2}) 2^{\nu_2/2}} w_2^{\frac{\nu_2}{2}-1} e^{-w_2/2} \mathbb{1}(w_1 > 0, w_2 > 0).$$

The support of (R, U) is $\{(r,u) : r > 0, u > 0\}$, and

$$\begin{aligned} R &= \frac{W_1/\nu_1}{W_2/\nu_2} \;\; (= g_1(w_1,w_2)) \\ U &= W_2 \;\; (= g_2(w_1,w_2)) \end{aligned} \iff \begin{aligned} W_1 &= RU\frac{\nu_1}{\nu_2} \;\; (= g_1^{-1}(r,u)) \\ W_2 &= U \;\; (= g_2^{-1}(r,u)), \end{aligned}$$

giving the Jacobian

$$J(r,u) = \begin{vmatrix} \frac{\partial}{\partial r} ru\frac{\nu_1}{\nu_2} & \frac{\partial}{\partial u} ru\frac{\nu_1}{\nu_2} \\ \frac{\partial}{\partial r} u & \frac{\partial}{\partial u} u \end{vmatrix} = \begin{vmatrix} u\frac{\nu_1}{\nu_2} & r\frac{\nu_1}{\nu_2} \\ 0 & 1 \end{vmatrix} = u\frac{\nu_1}{\nu_2}.$$

Now we get

$$f_{(R,U)}(r,u) = \frac{1}{\Gamma(\frac{\nu_1}{2}) 2^{\nu_1/2}} \left(ru\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}-1} e^{-\frac{(ru\frac{\nu_1}{\nu_2})}{2}} \frac{1}{\Gamma(\frac{\nu_2}{2}) 2^{\nu_2/2}} u^{\frac{\nu_2}{2}-1} e^{-\frac{u}{2}} \left|u\frac{\nu_1}{\nu_2}\right| \mathbb{1}(r > 0, u > 0).$$

Now we integrate $f_{(R,U)}$ over u to get f_R:

$$f_R(r) = \int_0^\infty \frac{1}{\Gamma\left(\frac{\nu_1}{2}\right)2^{\nu_1/2}}\left(ru\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}-1} e^{-\frac{\left(ru\frac{\nu_1}{\nu_2}\right)}{2}} \frac{1}{\Gamma\left(\frac{\nu_2}{2}\right)2^{\nu_2/2}} u^{\frac{\nu_2}{2}-1} e^{-\frac{u}{2}} \left|u\frac{\nu_1}{\nu_2}\right| du \, \mathbb{1}(r>0)$$

$$= \int_0^\infty \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \frac{r^{\frac{\nu_1}{2}-1}}{2^{\frac{\nu_1+\nu_2}{2}}} u^{\frac{\nu_1+\nu_2}{2}-1} e^{-\frac{u}{2}\left(1+r\frac{\nu_1}{\nu_2}\right)} du \quad \mathbb{1}(r>0)$$

$$= \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \frac{r^{\frac{\nu_1}{2}-1}\left[\frac{1}{2}\left(1+r\frac{\nu_1}{\nu_2}\right)\right]^{-\frac{\nu_1+\nu_2}{2}}}{2^{\frac{\nu_1+\nu_2}{2}}} \underbrace{\int_0^\infty \frac{u^{\frac{\nu_1+\nu_2}{2}} e^{-u/\left[\frac{1}{2}\left(1+r\frac{\nu_1}{\nu_2}\right)\right]^{-1}}}{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)\left[\frac{1}{2}\left(1+r\frac{\nu_1}{\nu_2}\right)\right]^{-\frac{\nu_1+\nu_2}{2}}} du}_{=1,\ \text{integral over pdf of } \mathrm{Gamma}\left(\frac{\nu_1+\nu_2}{2},\ \left[\frac{1}{2}\left(1+r\frac{\nu_1}{\nu_2}\right)\right]\right)} \mathbb{1}(r>0)$$

$$= \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{r^{\frac{\nu_1-2}{2}}}{\left(1+r\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1+\nu_2}{2}}} \mathbb{1}(r>0),$$

which is the pdf of the F_{ν_1,ν_2} distribution.

Proofs of III and IV of the theorem:

III. Proof of $\frac{\bar{X}_n-\mu}{S_n/\sqrt{n}} \sim t_{n-1}$:

We may write

$$\frac{\bar{X}_n-\mu}{S_n/\sqrt{n}} = \underbrace{\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}}}_{\sim \mathrm{Normal}(0,1),\text{ by I.}} \Big/ \sqrt{\underbrace{\frac{(n-1)S_n^2}{\sigma^2}}_{\text{A } \chi^2_{n-1} \text{ rv}}/(n-1)}$$

A χ^2_{n-1} rv divided by $n-1$

Since $\bar{X}_n$ and S_n^2 are independent, we have exactly the anatomy of a t_{n-1}-distributed rv. $\square$

IV. Proof of $\dfrac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1,n_2-1}$:

We may write

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{\frac{(n_1-1)S_1^2}{\sigma_1^2}\Big/(n_1-1)}{\frac{(n_2-1)S_2^2}{\sigma_2^2}\Big/(n_2-1)},$$

which is the ratio of two independent chi-square rvs, each divided by its degrees of freedom — the anatomy of an F_{n_1-1,n_2-1}-distributed rv. □

EXAMPLES OF USING THESE PIVOT QUANTITIES

I. Let $X_1,\dots,X_{16}$ be a random sample from the $\text{Normal}(\mu,5)$ distribution, where μ is unknown.

Then

$$\frac{\bar{X}_{16}-\mu}{\sqrt{5}/4} \sim \text{Normal}(0,1),$$

so that the interval with random endpoints

$$\bar{X}_{16} \pm z_{\alpha/2}\frac{\sqrt{5}}{4}$$

will contain μ with probability $1-\alpha$.

Having observed a realization of the random sample $X_1,\dots,X_{16}$ with $\bar{x}_{16}=2.3$, say, we construct a 95% C.I. for μ as

$$2.3 \pm \underbrace{1.96}_{=z_{.05/2} = \text{qnorm}(1-.05/2)}\frac{\sqrt{5}}{4}.$$

II. Let $X_1, \ldots, X_{16}$ be a random sample from the Normal(μ, σ^2) distribution, where μ and σ^2 are unknown.

Then $(16-1)S_{16}^2/\sigma^2 \sim \chi^2_{16-1}$, so that the interval with random endpoints

$$\left(\frac{(16-1)S_{16}^2}{\chi^2_{16-1,\alpha/2}} , \frac{(16-1)S_{16}^2}{\chi^2_{16-1,1-\alpha/2}} \right)$$

will contain σ^2 with probability $1-\alpha$.

Having observed a realization of the random sample $X_1, \ldots, X_{16}$ with $S_{16}^2 = 4.5$ say, we construct a 99% C.I. for σ^2 as

$$\left(\frac{(16-1)\,4.5}{\underbrace{32.801}} , \frac{(16-1)\,4.5}{\underbrace{4.601}} \right)$$

$\chi^2_{15,\frac{.01}{2}} = \text{qchisq}(.995, 15)$ $\quad \chi^2_{15,1-\frac{.01}{2}} = \text{qchisq}(.005, 15)$

III. Let $X_1, \ldots, X_{16}$ be a random sample from the Normal(μ, σ^2) distribution, where μ and σ^2 are unknown.

Then $(\bar{X}_{16} - \mu)/(S_{16}/4) \sim t_{16-1}$, so that the interval with random endpoints

$$\bar{X}_{16} \pm t_{16-1,\alpha/2} \frac{S_n}{4}$$

will contain μ with probability $1-\alpha$.

Having observed a realization of the random sample $X_1, \ldots, X_{16}$ with $\bar{X}_{16} = 2.3$ and $S_{16}^2 = 4.5$, say, we construct a 90% C.I. for μ as

$$2.3 \pm \underbrace{1.753}_{t_{15,\frac{.1}{2}} = \text{qt}(.95,\,15)} \frac{\sqrt{4.5}}{4}.$$

VI. Consider two independent random samples

$$X_1, \dots, X_{17} \sim \text{Normal}(\mu_1, \sigma_1^2)$$

$$Y_1, \dots, Y_{26} \sim \text{Normal}(\mu_2, \sigma_2^2)$$

with μ_1, μ_2, σ_1^2, and σ_2^2 unknown. We have

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{17-1, 26-1},$$

so that the interval with random endpoints

$$\left(\frac{S_2^2}{S_1^2} F_{17-1, 26-1, 1-\alpha/2} \, , \, \frac{S_2^2}{S_1^2} F_{17-1, 26-1, \alpha/2} \right)$$

will contain σ_2^2 / σ_1^2 with probability $1-\alpha$.

Having observed realizations of the random samples $X_1, \dots, X_{17}$ and $Y_1, \dots, Y_{26}$ with $S_1^2 = 3.7$ and $S_2^2 = 4.9$, say, we may construct a 95% C.I. for σ_2^2/σ_1^2 as

$$\left(\frac{4.9}{3.7} \underbrace{(.383)}_{F_{17-1,26-1,1-\frac{.05}{2}} = qf(.025,16,25)} \, , \, \frac{4.9}{3.7} \underbrace{(2.384)}_{F_{17-1,26-1,\frac{.05}{2}} = qf(.975,16,25)} \right).$$

A MISCELLANEOUS RESULT:

Result: Let $T \sim t_\nu$. Then $T^2 \sim F_{1,\nu}$.

Proof: If $T \sim t_\nu$, then we may characterize it as

$$T = Z / \sqrt{W/\nu},$$

where $Z \sim \text{Normal}(0,1)$ and $W \sim \chi^2_\nu$, Z, W independent.

Then $T^2 = \dfrac{Z^2/1}{W/\nu} \sim F_{1,\nu}$, because $Z^2 \sim \chi^2_1$. $\square$

← Ratio of independent χ^2_1 and χ^2_ν rvs, each divided by its degrees of freedom.

A NOTE ON ONE-SIDED CIs

A $(1-\alpha)^*100\%$ upper confidence limit (UCL) for a parameter θ is a r.v. U such that

$$P(\theta < U) = 1-\alpha.$$

A $(1-\alpha)^*100\%$ lower confidence limit (LCL) for a parameter θ is a r.v. L such that

$$P(L < \theta) = 1-\alpha.$$

For μ (σ known): Suppose $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Normal}(\mu, \sigma^2)$, σ known.

Then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0,1),$$

so that we may write

$$P\left(-z_\alpha < \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right) = 1-\alpha$$

$$\Rightarrow \quad P\left(\mu < \bar{X}_n + z_\alpha \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha,$$

giving that $\bar{X}_n + z_\alpha \frac{\sigma}{\sqrt{n}}$ is a $(1-\alpha)^*100\%$ UCL for μ.

Likewise $\bar{X}_n - z_\alpha \frac{\sigma}{\sqrt{n}}$ is a $(1-\alpha)^*100\%$ LCL for μ.

The intervals

$$\left(-\infty, \bar{X}_n + z_\alpha \frac{\sigma}{\sqrt{n}}\right) \quad \text{and} \quad \left(\bar{X}_n - z_\alpha \frac{\sigma}{\sqrt{n}}, \infty\right),$$

each of which contain μ with probability $1-\alpha$, are called one-sided C.I.s

For μ (σ unknown): Suppose $X_1,\dots,X_n \overset{iid}{\sim} \text{Normal}(\mu,\sigma^2)$, σ unknown.

Then

$$\frac{\bar{X}_n-\mu}{S_n/\sqrt{n}} \sim t_{n-1}$$

so that we may write

$$P\left(-t_{n-1,\alpha} < \frac{\bar{X}_n-\mu}{S_n/\sqrt{n}}\right) = 1-\alpha$$

$$\Rightarrow \quad P\left(\mu < \bar{X}_n + t_{n-1,\alpha}\frac{S_n}{\sqrt{n}}\right) = 1-\alpha,$$

giving that $\bar{X}_n + t_{n-1,\alpha}\frac{S_n}{\sqrt{n}}$ is a $(1-\alpha)^*100\%$ UCL for μ.

Likewise $\bar{X}_n - t_{n-1,\alpha}\frac{S_n}{\sqrt{n}}$ is a $(1-\alpha)^*100\%$ LCL for μ.

The one-sided C.I.s

$$\left(-\infty, \bar{X}_n + t_{n-1,\alpha}\frac{S_n}{\sqrt{n}}\right) \quad \text{and} \quad \left(\bar{X}_n - t_{n-1,\alpha}\frac{S_n}{\sqrt{n}}, \infty\right)$$

each contain μ with probability $1-\alpha$.

For σ^2: Suppose $X_1,\dots,X_n \overset{iid}{\sim} \text{Normal}(\mu,\sigma^2)$, σ unknown.

Then $\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$,

so that we may write

$$P\left(\chi^2_{n-1,1-\alpha} < \frac{(n-1)S_n^2}{\sigma^2}\right) = 1-\alpha$$

$$\Rightarrow \quad P\left(\sigma^2 < \frac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha}}\right) = 1-\alpha,$$

giving that $\frac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha}}$ is a $(1-\alpha)^*100\%$ UCL for σ^2.

Likewise $\frac{(n-1)S_n^2}{\chi^2_{n-1,\alpha}}$ is a $(1-\alpha)^*100\%$ LCL for σ^2.

The one-sided C.I.s

$$\left(-\infty, \frac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha}}\right) \quad \text{and} \quad \left(\frac{(n-1)S_n^2}{\chi^2_{n-1,\alpha}}, \infty\right)$$

each contain μ with probability $1-\alpha$.