

# STAT 512 su 2021 Lec 04 slides

## Pivot quantities and sampling from the Normal distribution

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

## Pivot quantity

A *pivot quantity* is a function of sample statistics and population parameters which has a known distribution.

We will motivate pivot quantities by their use in constructing confidence intervals.

## Confidence interval

A  $(1 - \alpha)100\%$  *confidence interval* for a parameter  $\theta$  is an interval  $(L, U)$ , where  $L$  and  $U$  are random variables such that  $P(L < \theta < U) = 1 - \alpha$ .

**Exercise:** If  $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$ , then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \sim \text{Normal}(0, 1).$$

Use this result to construct a  $(1 - \alpha)100\%$  confidence interval for  $\mu$ .

## Standard Normal distribution:

- The pdf of the  $\text{Normal}(0, 1)$  distribution is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad \text{for } z \in \mathbb{R}.$$

- Denote by  $\Phi$  the cdf:  $\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$  for  $z \in \mathbb{R}$ .
- mgf:  $M_Z(t) = e^{t^2/2}$
- Let  $z_\xi$  satisfy  $P(Z > z_\xi) = \xi$ , where  $Z \sim \text{Normal}(0, 1)$ .

## Chi-squared distributions:

- The pdf of the  $\chi_\nu^2$  distribution is given by

$$f_X(x; \nu) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}} x^{\nu/2-1} \exp\left[-\frac{x}{2}\right], \quad \text{for } x > 0.$$

- $\nu$  is called the *degrees of freedom*
- mgf:  $M_X(t) = (1 - 2t)^{-\nu/2}$  for  $t < 1/2$ .
- Let  $\chi_{\nu,\xi}^2$  satisfy  $P(W > \chi_{\nu,\xi}^2) = \xi$ , where  $W \sim \chi_\nu^2$ .

Let  $Z_1, \dots, Z_\nu \stackrel{\text{ind}}{\sim} \text{Normal}(0, 1)$ , then

$$W = Z_1^2 + \dots + Z_\nu^2 \sim \chi_\nu^2.$$

**Exercise:** Prove the above.

## t distributions:

- The pdf of the  $t_\nu$  distribution is given by

$$f_T(t; \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2} \quad \text{for } t \in \mathbb{R}.$$

- $\nu$  is called the *degrees of freedom*
- mgf: does not exist!
- Let  $t_{\nu,\xi}$  satisfy  $P(T > t_{\nu,\xi}) = \xi$ , where  $T \sim t_\nu$ .

Let  $Z$  and  $W$  be independent rvs such that  $Z \sim \text{Normal}(0, 1)$  and  $W \sim \chi_\nu^2$ , then

$$T = \frac{Z}{\sqrt{W/\nu}} \sim t_\nu.$$

**Exercise:** Prove above via finding the joint density of  $(T, U)$ , where  $U = W$ .

## F distributions:

- The pdf of the  $F_{\nu_1, \nu_2}$  distribution is given by

$$f_R(r; \nu_1, \nu_2) = \frac{\Gamma(\frac{\nu_1 + \nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} r^{(\nu_1-2)/2} \left(1 + \frac{\nu_1}{\nu_2} r\right)^{-(\nu_1 + \nu_2)/2}$$

for  $r > 0$ .

- $\nu_1$  and  $\nu_2$  are called the *numerator and denominator degrees of freedom*.
- mgf: does not exist!
- Let  $F_{\nu_1, \nu_2, \xi}$  satisfy  $P(R > F_{\nu_1, \nu_2, \xi}) = \xi$ , where  $R \sim F_{\nu_1, \nu_2, \xi}$ .

Let  $W_1$  and  $W_2$  be independent rvs such that  $W_1 \sim \chi_{\nu_1}^2$  and  $W_2 \sim \chi_{\nu_2}^2$ , then

$$R = \frac{W_1/\nu_1}{W_2/\nu_2} \sim F_{\nu_1, \nu_2}.$$

**Exercise:** Prove above via finding the joint density of  $(R, U)$ , where  $U = W_2$ .

## Theorem (Pivot quantity results with sample from Normal)

Let  $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$  and  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ ,  $S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ .

Then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0, 1), \quad \frac{(n-1)S_n^2}{\sigma^2} \sim \chi_{n-1}^2, \quad \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{n-1}$$

**Exercise:** Derive the above results and apply to confidence intervals.

## Theorem (Pivot quantity results with two Normal samples)

For two independent random samples

$$X_1, \dots, X_{n_1} \stackrel{\text{ind}}{\sim} \text{Normal}(\mu_1, \sigma_1^2), \quad \bar{X} = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i, \quad S_1^2 = \frac{1}{n_1-1} \sum_{i=1}^{n_1} (X_i - \bar{X})^2$$
$$Y_1, \dots, Y_{n_2} \stackrel{\text{ind}}{\sim} \text{Normal}(\mu_2, \sigma_2^2), \quad \bar{Y} = \frac{1}{n_2} \sum_{i=1}^{n_2} Y_i, \quad S_2^2 = \frac{1}{n_2-1} \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2$$

we have

$$\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim \text{Normal}(0, 1)$$
$$\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \stackrel{\text{approx}}{\sim} t_{\hat{\nu}} \quad (\text{discuss later})$$
$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1, n_2-1}$$

**Exercise:** Derive the above results and apply to confidence intervals.



## Exercises:

- ① Let  $X_1, \dots, X_{16} \stackrel{\text{ind}}{\sim} \text{Normal}(\mu, 5)$ ,  $\mu$  unkn,  $\bar{X}_{16} = 2.3$ . Give 95% CI for  $\mu$ .
- ② Let  $X_1, \dots, X_{16} \stackrel{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$ ,  $\mu, \sigma^2$  unkn,  $S_{16}^2 = 4.5$ . Give 95% CI for  $\sigma^2$ .
- ③ Let  $X_1, \dots, X_{16} \stackrel{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$ ,  $\mu, \sigma^2$  unkn,  $\bar{X}_{16} = 2.3$ ,  $S_{16}^2 = 4.5$ .  
Give 95% CI for  $\mu$ .

④ Let

$$X_1, \dots, X_{17} \stackrel{\text{ind}}{\sim} \text{Normal}(\mu_1, \sigma_1^2)$$

$$Y_1, \dots, Y_{26} \stackrel{\text{ind}}{\sim} \text{Normal}(\mu_2, \sigma_2^2),$$

$\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$  unkn,  $S_1^2 = 3.7$  and  $S_2^2 = 4.9$ . Give 95% CI for  $\sigma_2^2/\sigma_1^2$ .

## One-sided confidence intervals

A  $(1 - \alpha)100\%$  *upper confidence limit* for a parameter  $\theta$  is a rv  $U$  such that

$$P(\theta < U) = 1 - \alpha.$$

A  $(1 - \alpha)100\%$  *lower confidence limit* for a parameter  $\theta$  is a rv  $L$  such that

$$P(L < \theta) = 1 - \alpha.$$

**Example:** Show construction of one-sided CI for  $\mu$  in Normal,  $\sigma$ -known case.