STAT 512 su 2021 Lec 05 slides

Parametric estimation and properties of estimators

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Parametric framework:

Let $X_1, \ldots, X_n$ be rvs with joint dist. as a function of $(\theta_1, \ldots, \theta_d) \in \Theta$.

- The values $\theta_1, \ldots, \theta_d$ are called *parameters*.
- The set Θ is the *parameter space*.
- The number of parameters d is finite.
- If we know the values of $\theta_1, \ldots, \theta_d$, then we know everything.

Goal: Use $X_1, \ldots, X_n$ to estimate the values of $\theta_1, \ldots, \theta_d$.

Examples: Discuss Bernoulli, Normal, Exponential, SLR...

Nonparametric framework has parameters $\{\theta_k\}_{k=1}^{\infty}$, so that $d = \infty$...

Quality of estimators

We mostly focus on estimating a single parameter $\theta \in \Theta \subset \mathbb{R}$.

Bias of an estimator

The *bias* of an estimator $\hat{\theta}$ of $\theta \in \Theta \subset \mathbb{R}$ is defined as

$$\text{Bias}\,\hat{\theta} = \mathbb{E}\hat{\theta} - \theta.$$

If $\text{Bias}\,\hat{\theta} = 0$, we call $\hat{\theta}$ an *unbiased* estimator of θ.

Standard error of an estimator

The *standard error* of an estimator $\hat{\theta}$ of $\theta \in \Theta \subset \mathbb{R}$ is defined as

$$\text{SE}\,\hat{\theta} = \sqrt{\text{Var}\,\hat{\theta}}.$$

So the standard error of an estimator is just its standard deviation.

Mean squared error

The *mean squared error* of an estimator $\hat{\theta}$ of $\theta \in \Theta \subset \mathbb{R}$ is defined as

$$\text{MSE}\,\hat{\theta} = \mathbb{E}(\hat{\theta} - \theta)^2$$

MSE takes both bias and variance into account.

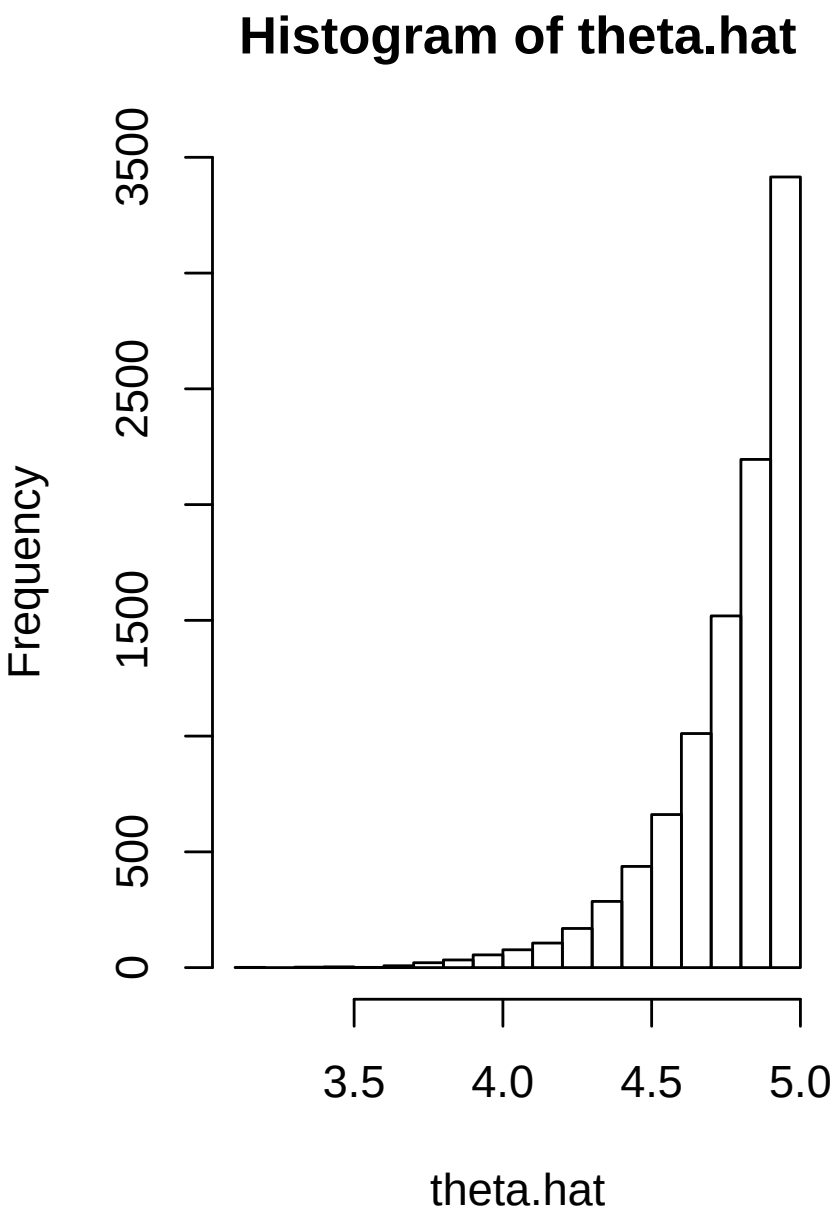
Exercise: Show that $\text{MSE}\,\hat{\theta} = \text{Var}\,\hat{\theta} + (\text{Bias}\,\hat{\theta})^2$.

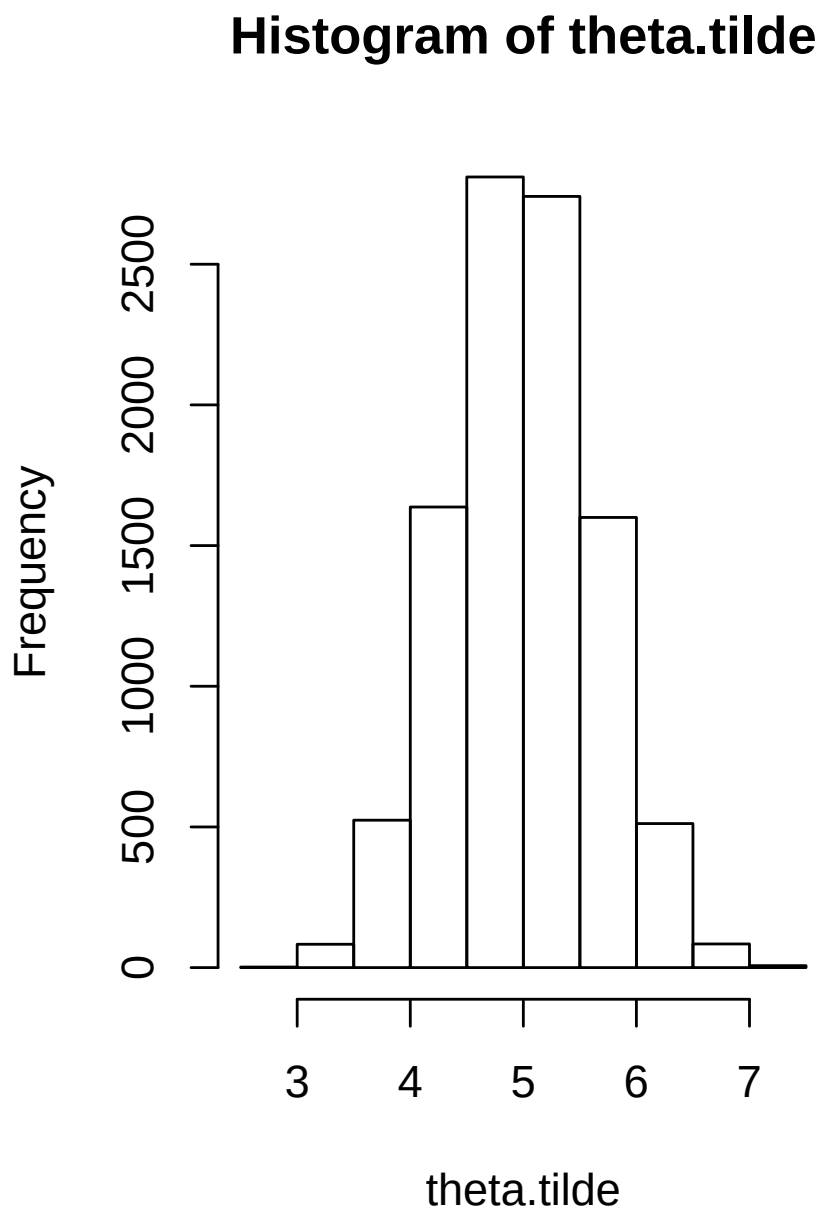
Exercise: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(0, \theta)$ and consider two estimators of θ:

$$\hat{\theta} = X_{(n)}$$
$$\tilde{\theta} = 2\bar{X}_n$$

Which estimator is better?

1. Find $\text{Bias}\,\hat{\theta}$ and $\text{Bias}\,\tilde{\theta}$.
2. Find $\text{Var}\,\hat{\theta}$ and $\text{Var}\,\tilde{\theta}$.
3. Compare $\text{MSE}\,\hat{\theta}$ and $\text{MSE}\,\tilde{\theta}$ at different sample sizes $n = 1, 2, \ldots$
4. Run a simulation to demonstrate $\text{MSE}\,\tilde{\theta}$ and $\text{MSE}\,\hat{\theta}$.
5. Propose a bias-corrected version $\hat{\theta}_{\text{unbiased}}$ of $\hat{\theta} = X_{(n)}$ and find $\text{MSE}\,\hat{\theta}_{\text{unbiased}}$.


Histogram of theta.hat
Frequency
0
500
1500
2500
3500
3.5
4.0
4.5
5.0
theta.hat


Histogram of theta.tilde
Frequency
0
500
1000
1500
2000
2500
3
4
5
6
7
theta.tilde

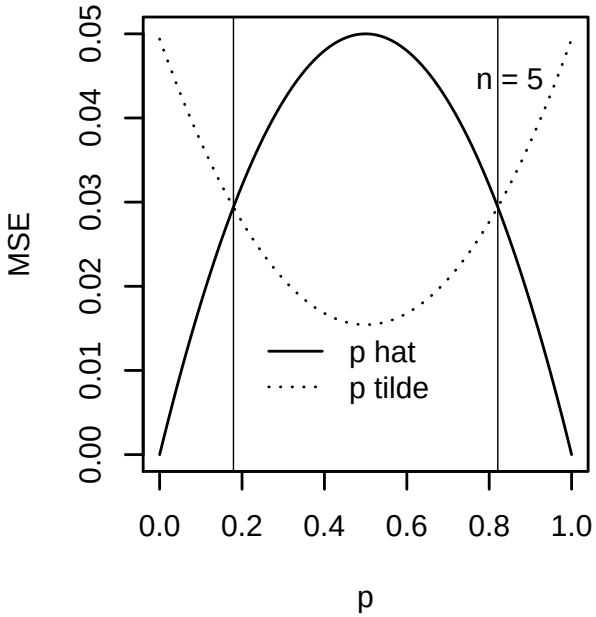
Exercise: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$ and consider two estimators of p:

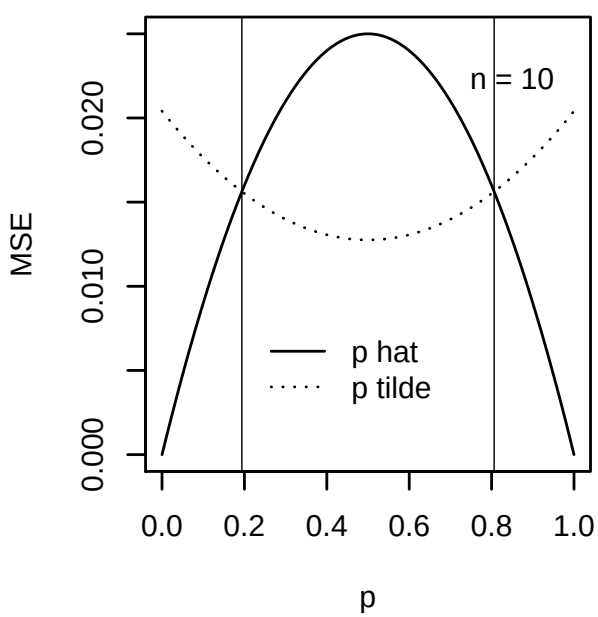
$$\hat{p} = \frac{Y}{n}$$

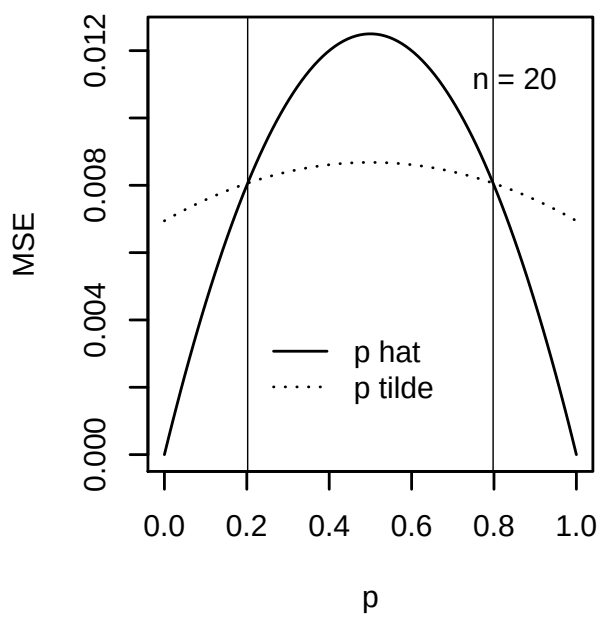
$$\tilde{p} = \frac{Y+2}{n+4}, \qquad \text{where } Y = X_1 + \cdots + X_n.$$

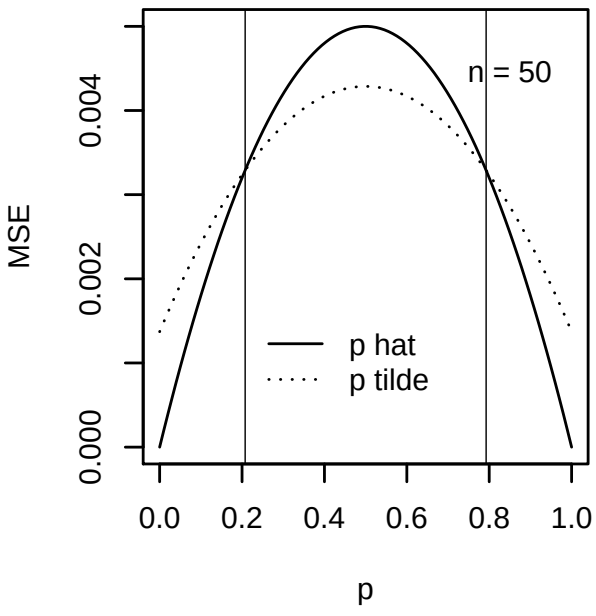
Which estimator is better?

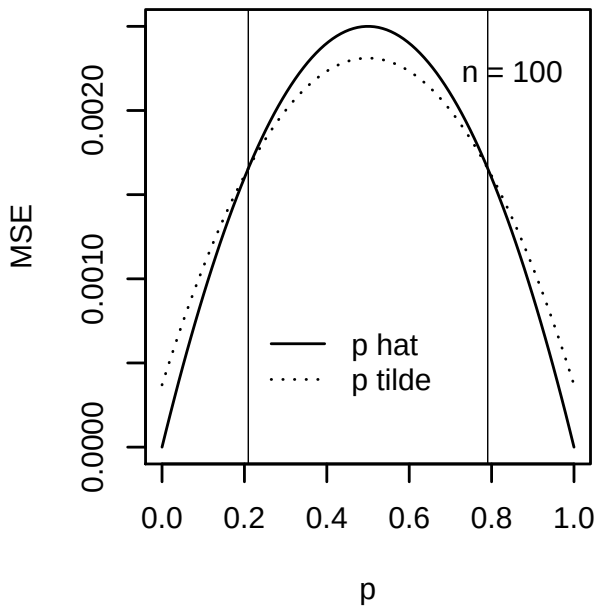
1. Find Bias $\hat{p}$ and Bias $\tilde{p}$.
2. Find Var $\hat{p}$ and Var $\tilde{p}$.
3. Compare MSE $\hat{p}$ and MSE $\tilde{p}$ at different values of p (see R supplement).

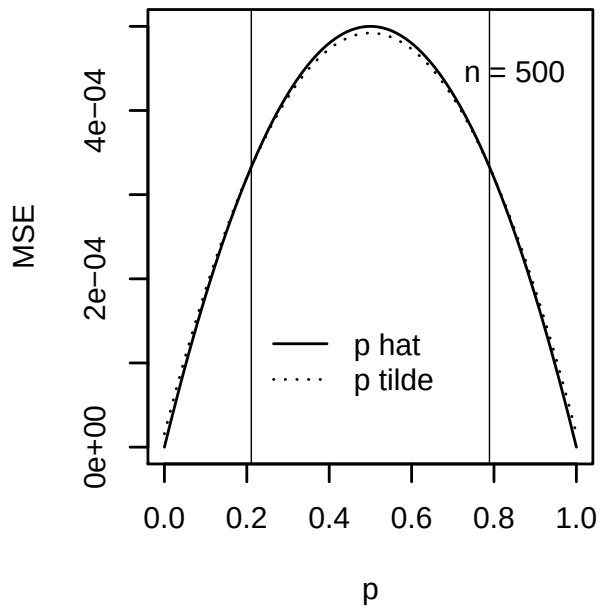

MSE
0.00
0.01
0.02
0.03
0.04
0.05
n = 5
p hat
p tilde
0.0
0.2
0.4
0.6
0.8
1.0
p


MSE
0.000
0.010
0.020
n = 10
p hat
p tilde
0.0
0.2
0.4
0.6
0.8
1.0
p


MSE
0.000
0.004
0.008
0.012
n = 20
p hat
p tilde
0.0
0.2
0.4
0.6
0.8
1.0
p


MSE
0.000
0.002
0.004
n = 50
p hat
p tilde
0.0
0.2
0.4
0.6
0.8
1.0
p


MSE
0.0000
0.0010
0.0020
n = 100
p hat
p tilde
0.0
0.2
0.4
0.6
0.8
1.0
p


MSE
0e+00
2e−04
4e−04
n = 500
p hat
p tilde
0.0
0.2
0.4
0.6
0.8
1.0
p

Suppose we want to estimate a function $\tau = \tau(\theta)$ of θ using an unbiased estimator $\hat{\theta}$.

$$\mathbb{E}\hat{\theta} = \theta \quad \nRightarrow \quad \mathbb{E}\tau(\hat{\theta}) = \tau(\theta).$$

Exercise: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Exponential}(\lambda)$ and set $\hat{\lambda} = n^{-1}(X_1 + \cdots + X_n)$.

1. Show that $\hat{\lambda}$ is an unbiased estimator of λ.
2. Consider using $\hat{\tau} = 1/\hat{\lambda}$ as an estimator of $\tau = 1/\lambda$. Is it unbiased?
3. Propose an unbiased estimator of τ.