STAT 512 su 2021 Lec 06 slides

Large-sample properties of estimators: consistency and the WLLN

Karl B. Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Consistency of an estimator

An estimator $\hat{\theta}_n$ of $\theta \in \Theta \subset \mathbb{R}$ is called *consistent* if

$$\lim_{n\to\infty} P(|\hat{\theta}_n - \theta| < \varepsilon) = 1$$

for every $\varepsilon > 0$ and every $\theta \in \Theta$.

Means the event $\hat{\theta}_n \in (\theta - \varepsilon, \theta + \varepsilon)$ occurs w/ prob $\to 1$ as $n \to \infty$, for any $\varepsilon > 0$.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Unif}(0, \theta)$. Is $X_{(n)}$ a consistent estimator of θ?

WLLN: Sample mean is consistent for pop. mean if pop. variance is finite.

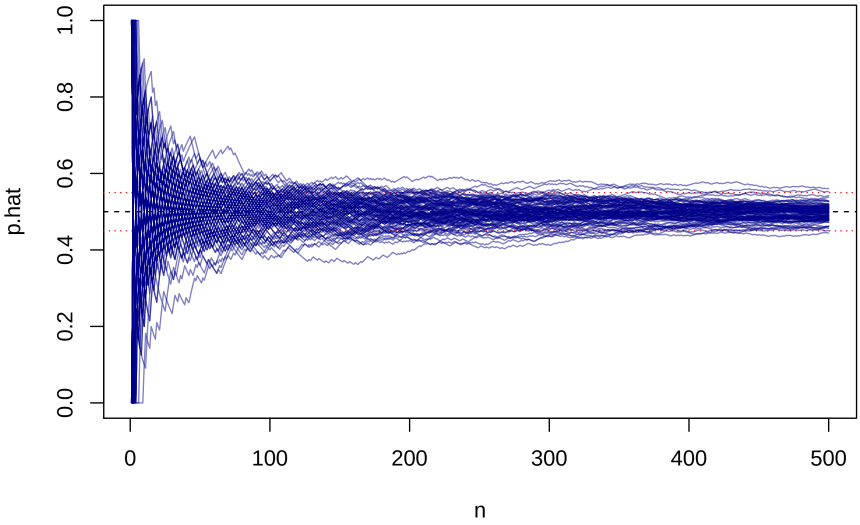
Theorem (Weak law of large numbers)

Let $X_1, \ldots, X_n$ be a rs from a dist. with mean μ and variance $\sigma^2 < \infty$. Then

$$\lim_{n \to \infty} P(|\bar{X}_n - \mu| < \varepsilon) = 1$$

for every $\varepsilon > 0$ and every $\mu \in \mathbb{R}$.

Exercise: Prove using Чебышёв's inequality.


p.hat
0.0
0.2
0.4
0.6
0.8
1.0
0
100
200
300
400
500
n

Theorem (Sufficient conditions for consistency)

An estimator $\hat{\theta}_n$ is consistent for θ if

1. $\lim_{n\to\infty} \operatorname{Var}\hat{\theta}_n = 0$
2. $\lim_{n\to\infty} \operatorname{Bias}\hat{\theta}_n = 0$

Note that $\operatorname{MSE}\hat{\theta}_n \to 0$ implies that $\hat{\theta}_n$ is consistent for θ.

Instead of showing $\lim_{n\to\infty} P(|\hat{\theta}_n - \theta| < \varepsilon) = 1$, can show $\operatorname{MSE}\hat{\theta}_n \to 0$ (easier!!!)

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(0, \theta)$. Check consistency of the estimators

$$\hat{\theta}_n = X_{(n)} \quad \text{and} \quad \tilde{\theta}_n = 2\bar{X}_n.$$

Exercise: Let $Y_1, \dots, Y_n \overset{\text{ind}}{\sim} \text{Exponential}(\lambda)$ and consider $\tilde{\lambda}_n = nY_{(1)}$.

1. Compute $\text{Bias}\, \tilde{\lambda}_n$
2. Compute $\text{Var}\, \tilde{\lambda}_n$
3. Is $\tilde{\lambda}_n$ a consistent estimator of λ?

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$. Let $Y = X_1 + \cdots + X_n$ and consider

$$\hat{p}_n = \frac{Y}{n} \quad \text{and} \quad \tilde{p}_n = \frac{Y+2}{n+4}.$$

Check the consistency of these estimators (MSEs derived in previous lecture).

Notation: If $\hat{\theta}_n$ is consistent for θ, we can write $\hat{\theta}_n \xrightarrow{\text{p}} \theta$.

✍️

The notation $\xrightarrow{\text{p}}$ stands more generally for *convergence in probability*.

Theorem (Helper results for proving consistency)

Let $\hat{\theta}_{1,n}$ and $\hat{\theta}_{2,n}$ be consistent estimators for θ_1 and θ_2, respectively. Then

1. $\hat{\theta}_{1,n} \pm \hat{\theta}_{2,n} \xrightarrow{p} \theta_1 \pm \theta_2$
2. $\hat{\theta}_{1,n}\hat{\theta}_{2,n} \xrightarrow{p} \theta_1\theta_2$
3. $\hat{\theta}_{1,n}/\hat{\theta}_{2,n} \xrightarrow{p} \theta_1/\theta_2$, *provided* $\theta_2 \neq 0$.
4. *For any continuous function* $\tau : \mathbb{R} \to \mathbb{R}$, $\tau(\hat{\theta}_{1,n}) \xrightarrow{p} \tau(\theta_1)$.
5. *For any sequences* $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$ *s.t.* $\lim_{n\to\infty} a_n = 1$ *and* $\lim_{n\to\infty} b_n \to 0$, *we have*
$$a_n\hat{\theta}_{1,n} + b_n \xrightarrow{p} \theta_{1,n}.$$

Exercises:

1. Show consistency of S_n^2 for population variance σ^2 when $\mu_4 < \infty$.
2. Show consistency of $\hat{p}(1-\hat{p})$ for $p(1-p)$ in Bernoulli(p) setting.