STAT 512 sp 2020 hw 1

1. Consider a function $g(x) = \sin(x)$ for $x \in (-\pi, \pi]$.
   (a) Give the set $\mathcal{Y} = \{y : g(x) = y \text{ for some } x \in (-\pi, \pi]\}$.
   (b) State whether g is a one-to-one function.
   (c) Give $g^{-1}([0, 1])$.
   (d) Give $g^{-1}(1/2)$.
   (e) Give $g^{-1}(\{-1, 1\})$.
2. Let $g(x) = e^x/(1 + e^x)$ for $x \in \mathbb{R}$.
   (a) Give the set $\mathcal{Y} = \{y : g(x) = y \text{ for some } x \in \mathbb{R}\}$.
   (b) State whether g is a one-to-one function.
   (c) Find an expression for $g^{-1}(y)$ for all $y \in \mathcal{Y}$.
3. Let $X \sim f_X(x) = (3/2)x^2\mathbf{1}(-1 \leq x \leq 1)$.
   (a) Find the pdf of $Y = 2X$. Be sure to note the support of Y.
   (b) Find the pdf of $Y = X + 1$. Be sure to note the support of Y.
   (c) Find the pdf of $Y = |X|$. Be sure to note the support of Y.
4. Let $X \sim \text{Gamma}(\alpha, \beta)$.
   (a) For some constant $c > 0$, identify the distribution of $Y = cX$.
   (b) The Gamma distribution parameter β is often called the *scale parameter*. Explain why the parameter has this name.
5. Let Y be a random variable with cdf given by

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ (y/a)^b, & 0 \leq y \leq a \\ 1, & y > a \end{cases}$$

   for some $a, b > 0$.
   (a) Find the pdf of Y.
   (b) We can generate a realization of Y by passing a $\text{Uniform}(0, 1)$ random variable through the quantile function of Y. Find the quantile function of Y. *Hint: Since the cdf is continuous, the quantile function is just the inverse function of the cdf.*
6. Let $X \sim \text{Uniform}(0, 1)$. Find the pdf of $Y = -\lambda \log X$, where $\lambda > 0$.
7. Let X be a rv with the $\text{Weibull}(k, \lambda)$ distribution, which has pdf

$$f(x) = \begin{cases} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

   for $k, \lambda > 0$.

(a) Find the pdf of $V = X/\lambda$.

(b) Find the pdf of $Z = X^k$.

(c) Find $\mathbb{E}X^k$.

(d) Let $Y \sim \text{Exponential}(1)$. For any $k > 0$, find the pdf of $W = \lambda Y^{1/k}$.

(e) Explain how you could transform a $\text{Uniform}(0, 1)$ realization into a $\text{Weibull}(k, \lambda)$ realization. *Hint: First transform the* $\text{Uniform}(0, 1)$ *to an* $\text{Exponential}(1)$*, and then make another transformation to get the* $\text{Weibull}(k, \lambda)$*.*

8. Let $X \sim \text{Uniform}(-\pi/2, \pi/2)$.

(a) Find the pdf f_Y of $Y = \tan(X)$.

(b) The pdf of the t_ν distribution, where $\nu > 0$ is the degrees of freedom, is given by

$$f_T(t; \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2} \qquad \text{for } -\infty < t < \infty.$$

Verify that the pdf of the t_1 distribution is the same as the answer to part (a).

9. Let T have the t_ν distribution, of which the pdf is given in Question 8. The t distributions will be very important later on this semester.

(a) Find the pdf of $R = T^2$.

(b) The pdf of the F_{ν_1,ν_2} distribution, where $\nu_1 > 0$ and $\nu_2 > 0$ are called, respectively, the numerator and the denominator degrees of freedom, is given by

$$f_R(r; \nu_1, \nu_2) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} r^{(\nu_1-2)/2} \left(1 + \frac{\nu_1}{\nu_2} r\right)^{-(\nu_1+\nu_2)/2} \mathbf{1}(r > 0).$$

The F distributions will be very important later on this semester. Argue that squaring a t_ν random variable results in a $F_{1,\nu}$ random variable.

10. Use R to generate 100 realizations of the random variable

$$X \sim f_X(x) = 0.2e^{-0.2x}\mathbf{1}(x > 0)$$

by generating $\text{Uniform}(0, 1)$ realizations and transforming them. Turn in the following:

(a) R code.

(b) A histogram of the 100 realizations of X.