STAT 512 su 2021 hw 1

1. Consider a function $g(x) = \sin(x)$ for $x \in (-\pi, \pi]$.
   (a) Give the set $\mathcal{Y} = \{y : g(x) = y \text{ for some } x \in (-\pi, \pi]\}$.

   $\mathcal{Y} = [-1, 1]$.

   (b) State whether g is a one-to-one function.

   It is not a one-to-one function, because, $\sin(\pi/6) = \sin(5\pi/6) = 1/2$, for example.

   (c) Give $g^{-1}([0, 1])$.

   $g^{-1}([0, 1]) = [0, \pi]$.

   (d) Give $g^{-1}(1/2)$.

   $g^{-1}(1/2) = \{\pi/6, 5\pi/6\}$.

   (e) Give $g^{-1}(\{-1, 1\})$.

   $g^{-1}(\{-1, 1\}) = \{-\pi/2, \pi/2\}$.

2. Let $g(x) = e^x/(1 + e^x)$ for $x \in \mathbb{R}$.
   (a) Give the set $\mathcal{Y} = \{y : g(x) = y \text{ for some } x \in \mathbb{R}\}$.

   $\mathcal{Y} = (0, 1)$.

   (b) State whether g is a one-to-one function.

   Yes, it is one-to-one.

   (c) Find an expression for $g^{-1}(y)$ for all $y \in \mathcal{Y}$.

   $g^{-1}(y) = \log(y/(1 - y))$.

3. Let $X \sim f_X(x) = (3/2)x^2\mathbf{1}(-1 \le x \le 1)$.
   (a) Find the pdf of $Y = 2X$. Be sure to note the support of Y.

We have

$$y = 2x =: g(x) \iff x = y/2 =: g^{-1}(y), \quad \text{and} \quad \frac{d}{dy}g^{-1}(y) = 1/2.$$

By the transformation method we have

$$f_Y(y) = (3/2)(y/2)^2\mathbf{1}(-1 \le y/2 \le 1)|1/2| = (3/16)y^2\mathbf{1}(-2 \le y \le 2).$$

(b) Find the pdf of $Y = X + 1$. Be sure to note the support of Y.

We have

$$y = x + 1 =: g(x) \iff x = y - 1 =: g^{-1}(y), \quad \text{and} \quad \frac{d}{dy}g^{-1}(y) = 1.$$

By the transformation method we have

$$f_Y(y) = (3/2)(y-1)^2\mathbf{1}(-1 \le y - 1 \le 1)|1| = (3/2)(y-1)^2\mathbf{1}(0 \le y \le 2).$$

(c) Find the pdf of $Y = |X|$. Be sure to note the support of Y.

Since the transformation is not monotone, we cannot use the transformation method. Let's use the cdf method. We have

$$F_Y(y) = P(Y \le y) = P(|X| \le y) = P(-y \le X \le y) = \int_{-y}^{y}(3/2)x^2\mathbf{1}(-1 \le x \le 1)dx.$$

For $y \in [0,1]$, we have

$$\int_{-y}^{y}(3/2)x^2\underbrace{\mathbf{1}(-1 \le x \le 1)}_{=1 \text{ when } y \in [0,1]}dx = \int_{-y}^{y}(3/2)x^2dx = x^3/2\Big|_{-y}^{y} = y^3.$$

So the cdf is given by

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ y^3, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$

Taking the derivative of each piece separately, we obtain

$$\begin{aligned} f_Y(y) &= \begin{cases} 0, & y < 0 \\ 3y^2, & 0 \le y \le 1 \\ 0, & y > 1 \end{cases} \\ &= 3y^2\mathbf{1}(0 \le y \le 1). \end{aligned}$$

4. Let $X \sim \text{Gamma}(\alpha, \beta)$.

(a) For some constant $c > 0$, identify the distribution of $Y = cX$.

We have

$$y = cx =: g(x) \iff x = y/c =: g^{-1}(y), \quad \text{and} \quad \frac{d}{dy} g^{-1}(y) = 1/c.$$

The support of Y is $(0, \infty)$. By the transformation method we have

$$\begin{aligned} f_Y(y) &= \frac{1}{\Gamma(\alpha)\beta^\alpha}(y/c)^{\alpha-1} \exp(-(y/c)/\beta)|1/c| \\ &= \frac{1}{\Gamma(\alpha)(c\beta)^\alpha} y^{\alpha-1} \exp(-y/(c\beta)) \end{aligned}$$

for $y > 0$, so $Y \sim \text{Gamma}(\alpha, c\beta)$.

(b) The Gamma distribution parameter β is often called the *scale parameter*. Explain why the parameter has this name.

If you scale a $\text{Gamma}(\alpha, \beta)$ rv by a constant c, the result is a $\text{Gamma}(\alpha, c\beta)$ rv, so that the parameter β is scaled by c.

5. Let Y be a random variable with cdf given by

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ (y/a)^b, & 0 \le y \le a \\ 1, & y > a \end{cases}$$

for some $a, b > 0$.

(a) Find the pdf of Y.

Taking the derivative of each piece separately, we obtain

$$\begin{aligned} f_Y(y) &= \begin{cases} 0, & y < 0 \\ (b/a)(y/a)^{b-1}, & 0 \le y \le a \\ 0, & y > a \end{cases} \\ &= (b/a)(y/a)^{b-1}\mathbf{1}(0 \le y \le a). \end{aligned}$$

(b) We can generate a realization of Y by passing a $\text{Uniform}(0, 1)$ random variable through the quantile function of Y. Find the quantile function of Y. *Hint: Since the cdf is continuous, the quantile function is just the inverse function of the cdf.*

We write

$$u = (y/a)^b \iff au^{1/b} = y,$$

so the function $F_Y^{-1}(u) = au^{1/b}$ returns the uth quantile of Y for $u \in [0,1]$. Therefore, if we generate a Uniform$(0,1)$ realization U, then $Y = aU^{1/b}$ will be a realization from the distribution with cdf F_Y.

6. Let $X \sim \text{Uniform}(0,1)$. Find the pdf of $Y = -\lambda \log X$, where $\lambda > 0$.

We have $f_X(x) = \mathbf{1}(0 < x < 1)$ and

$$y = -\lambda \log x \iff x = e^{-y/\lambda}, \quad \text{and} \quad \frac{d}{dy} e^{-y/\lambda} = (1/\lambda) e^{-y/\lambda}.$$

The support of Y is $(0, \infty)$. By the transformation method we have

$$f_Y(y) = \frac{1}{\lambda} e^{-y/\lambda}$$

for $y > 0$.

7. Let X be a rv with the Weibull(k, λ) distribution, which has pdf

$$f(x) = \begin{cases} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

for $k, \lambda > 0$.

(a) Find the pdf of $V = X/\lambda$.

We have

$$v = x/\lambda =: g(x) \iff x = \lambda v =: g^{-1}(v), \quad \text{and} \quad \frac{d}{dv} g^{-1}(v) = \lambda.$$

The support of V is $(0, \infty)$. By the transformation method we have

$$f_V(v) = \frac{k}{\lambda}\left(\frac{\lambda v}{\lambda}\right)^{k-1} e^{-(\lambda v/\lambda)^k} |\lambda| = k v^{k-1} e^{-v^k}$$

for $v > 0$.

(b) Find the pdf of $Z = X^k$.

We have

$$z = x^k =: g(x) \iff x = z^{1/k} =: g^{-1}(z) \quad \text{and} \quad \frac{d}{dz}g^{-1}(z) = (1/k)z^{1/k-1}.$$

The support of Z is $(0, \infty)$. By the transformation method we have

$$f_Z(z) = \frac{k}{\lambda}\left(\frac{z^{1/k}}{\lambda}\right)^{k-1} e^{-(z^{1/k}/\lambda)^k}|(1/k)z^{1/k-1}| = (1/\lambda^k)e^{-z/\lambda^k}$$

for $z > 0$.

(c) Find $\mathbb{E}X^k$.

By the previous result, we see that $\mathbb{E}X^k = \lambda^k$, since X^k has the Exponential(λ^k) distribution, which has mean λ^k.

(d) Let $Y \sim$ Exponential(1). For any $k > 0$, find the pdf of $W = \lambda Y^{1/k}$.

We have $f_Y(y) = e^{-y}$ for $y > 0$ and

$$w = \lambda y^{1/k} =: g(y) \iff y = (w/\lambda)^k =: g^{-1}(w), \quad \text{and} \quad \frac{d}{dw}g^{-1}(w) = (k/\lambda^k)w^{k-1}.$$

The support of W is $(0, \infty)$. By the transformation method we have

$$f_V(v) = \exp(-(w/\lambda)^k)|(k/\lambda^k)w^{k-1}| = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}$$

for $w > 0$, which is the pdf of the Weibull(k, λ) distribution.

(e) Explain how you could transform a Uniform$(0, 1)$ realization into a Weibull(k, λ) realization. *Hint: First transform the Uniform$(0, 1)$ to an Exponential(1), and then make another transformation to get the Weibull(k, λ).*

The transformation is $\lambda(-\log U)^{1/k}$, since $-\log U \sim$ Exponential(1), by the answer to Question 6. We can also use $-\log(1 - U) \sim$ Exponential(1).

8. Let $X \sim$ Uniform$(-\pi/2, \pi/2)$.

(a) Find the pdf f_Y of $Y = \tan(X)$.

We have $f_X(x) = (1/\pi)\mathbf{1}(-\pi/2 < x < \pi/2)$ and

$$y = \tan(x) =: g(x) \iff x = \tan^{-1}(y) \quad \text{and} \quad \frac{d}{dy}\tan^{-1}(y) = 1/(1 + y^2).$$

The support of Y is $(-\infty, \infty)$. By the transformation method we have

$$f_Y(y) = \frac{1}{\pi}\frac{1}{1+y^2}$$

for all $y \in \mathbb{R}$. This is the pdf of the Cauchy distribution!

(b) The pdf of the t_ν distribution, where $\nu > 0$ is the degrees of freedom, is given by

$$f_T(t;\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{\sqrt{\nu\pi}}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \quad \text{for } -\infty < t < \infty.$$

Verify that the pdf of the t_1 distribution is the same as the answer to part (a).

Use $\Gamma(1/2) = \sqrt{\pi}$ to show that

$$f_T(t;1) = \frac{1}{\pi}\frac{1}{1+t^2} \quad \text{for } -\infty < t < \infty.$$

9. Let T have the t_ν distribution, of which the pdf is given in Question 8. The t distributions will be very important later on this semester.

(a) Find the pdf of $R = T^2$.

The transformation is not monotone over the support of T, so we cannot use the transformation method; it is not a shift-and-scale transformtion, we cannot use the mgf method; let's use the cdf method. We write

$$F_R(r) = P(R \leq r) = P(T^2 < r) = P(-\sqrt{r} < T < \sqrt{r}) = F_T(\sqrt{r};\nu) - F_T(-\sqrt{r};\nu)$$

for $r > 0$, where $F_T(\cdot;\nu)$ is the cdf of the t_ν distribution. Taking the derivative of the above with respect to r gives

$$f_R(r) = f_T(\sqrt{r};\nu)\left(\frac{1}{2\sqrt{r}}\right) - f_T(-\sqrt{r};\nu)\left(-\frac{1}{2\sqrt{r}}\right) = f_T(\sqrt{r};\nu)/\sqrt{r},$$

since $f_T(a;\nu) = f_T(-a;\nu)$ for all $a \in \mathbb{R}$. So we have

$$f_R(r) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{\sqrt{\nu\pi}}\frac{1}{\sqrt{r}}\left(1+\frac{r}{\nu}\right)^{-(\nu+1)/2} \quad \text{for } -\infty < r < \infty.$$

(b) The pdf of the F_{ν_1,ν_2} distribution, where $\nu_1 > 0$ and $\nu_2 > 0$ are called, respectively, the numerator and the denominator degrees of freedom, is given by

$$f_R(r;\nu_1,\nu_2) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} r^{(\nu_1-2)/2}\left(1+\frac{\nu_1}{\nu_2}r\right)^{-(\nu_1+\nu_2)/2}\mathbf{1}(r>0).$$

The F distributions will be very important later on this semester. Argue that squaring a t_ν random variable results in a $F_{1,\nu}$ random variable.

We see that the pdf of the $F_{1,\nu}$ distribution can be simplified to the answer from part (a).

10. Use R to generate 100 realizations of the random variable

$$X \sim f_X(x) = 0.2e^{-0.2x}\mathbf{1}(x>0)$$

by generating $\text{Uniform}(0,1)$ realizations and transforming them. Turn in the following:

(a) R code.

The rv X has the $\text{Exponential}(5)$ distribution, which we can see by writing its pdf as

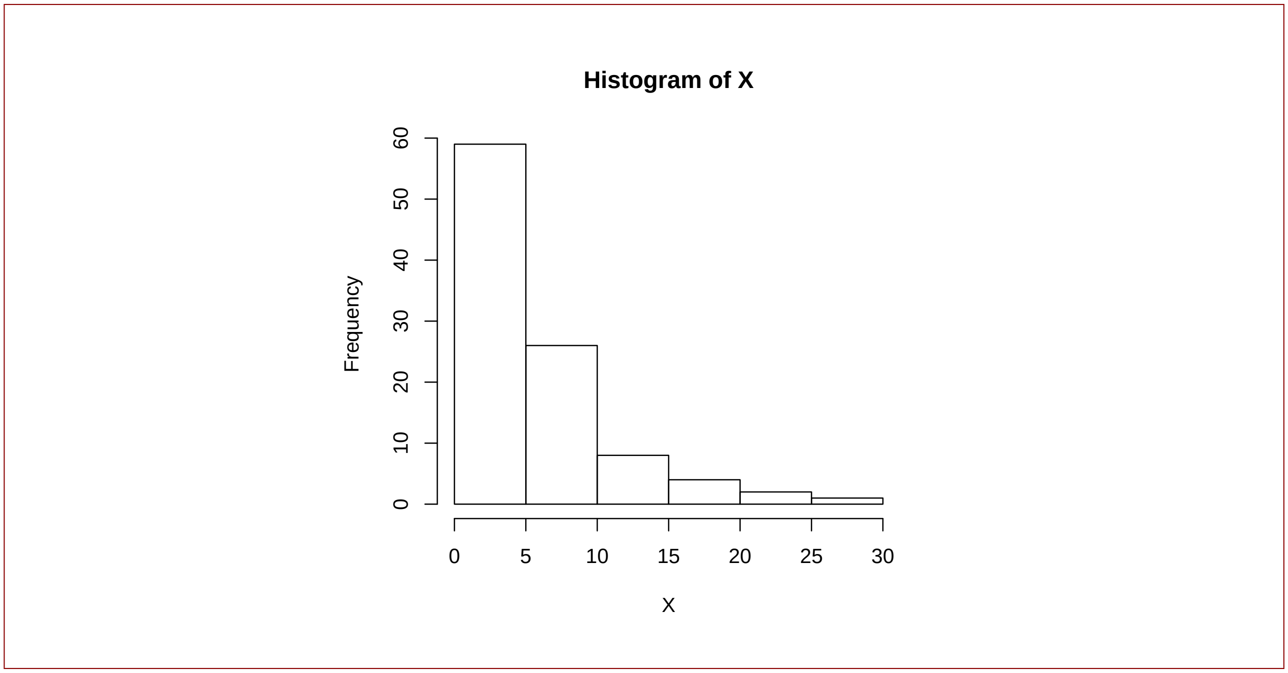
$$f_X(x) = \frac{1}{1/0.2}e^{-x/(1/0.2)}\mathbf{1}(x>0) = \frac{1}{5}e^{-x/5}\mathbf{1}(x>0).$$

According to the answer to Question 6, we need to generate 100 realizations from the $\text{Uniform}(0,1)$ distribution and transform them according to $X = -5\log U$. The following code does this:

```
U <- runif(100)
X <- -5*log(U)
```

(b) A histogram of the 100 realizations of X.

The histogram should like something like this:


Histogram of X
Frequency
0
10
20
30
40
50
60
0
5
10
15
20
25
30
X