STAT 512 su 2021 hw 2

1. Let U_1 and U_2 be independent $\text{Uniform}(0,1)$ random variables and let $Y = U_1U_2$.
   (a) Write down the joint pdf of U_1 and U_2.
   (b) Find the cdf of Y by obtaining an expression for $F_Y(y) = P(Y \leq y) = P(U_1U_2 \leq y)$ for all y.
   (c) Find the pdf of Y by taking the derivative of $F_Y(y)$ with respect to y.
   (d) Let $X = U_2$ and find the joint pdf of the rv pair (X, Y) using the bivariate transformation method. Be careful when defining the joint support of (X, Y).
   (e) Integrate the joint pdf of (X, Y) over X in order to get the pdf of Y.

2. Let X_1 and X_2 be independent $\text{Exponential}(1)$ rvs.
   (a) Find the joint density of $U_1 = X_1$ and $U_2 = \log(X_1 + X_2) - \log(X_1)$.
   (b) Show that $U_2 \sim \text{Exponential}(1)$.
   (c) Tell whether U_1 and U_2 are independent.

3. Let $G_1 \sim \text{Gamma}(\alpha_1, \beta)$ and $G_2 \sim \text{Gamma}(\alpha_2, \beta)$ and let G_1 and G_2 be independent. Define $B_1 = G_1/(G_1 + G_2)$ and $B_2 = G_1 + G_2$.
   (a) Find the joint pdf of (B_1, B_2).
   (b) Check whether B_1 and B_2 are independent.
   (c) Give the marginal pdf of B_1 and identify its distribution.
   (d) Give the marginal pdf of B_2 and identify its distribution.

4. Let $X_1, \dots, X_N$ be independent rvs and $n_1, \dots, n_N$ be positive integers such that $X_i \sim \text{Binomial}(n_i, p)$ for $i = 1, \dots, N$. Give the pmf of $Y = X_1 + \cdots + X_N$.

5. Let $X_1, \dots, X_N$ be independent rvs and $\lambda_1, \dots, \lambda_n$ be positive real numbers such that $X_i \sim \text{Poisson}(\lambda_i)$ for $i = 1, \dots, n$. Give the pmf of $Y = X_1 + \cdots + X_n$.

6. Let $X_1, \dots, X_{25}$ be independent $\text{Normal}(\mu = 1, \sigma^2 = 5)$ rvs. Find the distributions of the following:
   (a) $Y_{25} = X_1 + \cdots + X_{25}$
   (b) $\bar{X}_{25} = \frac{1}{25}(X_1 + \cdots + X_{25})$

7. Let Y_1 and Y_2 be independent rvs such that $Y_1 \sim \text{Normal}(\mu_1, \sigma_1^2)$ and $Y_2 \sim \text{Normal}(\mu_2, \sigma_2^2)$ and let $a_1, a_2 \in \mathbb{R}$. Find the distribution of the random variable $a_1Y_1 + a_2Y_2$ using mgfs.

8. Suppose (Z_1, Z_2) are standard bivariate Normal rvs with correlation ρ, with joint pdf

$$f_{(Z_1,Z_2)}(z_1, z_2) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2}\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{1-\rho^2}\right].$$

   Choose a value of ρ between 0.5 and 0.9 and use R to generate 1,000 realizations of (Z_1, Z_2). Then transform these into realizations of $U_1 = Z_1 + Z_2$ and $U_2 = Z_1 - Z_2$. Then make two scatter plots: one of the Z_2 values against the Z_1 values and one of the U_2 values against the U_1 values. Turn in your code and these two plots, and say whether you think U_1 and U_2 are independent.

   The following code will generate the (Z_1, Z_2) realizations (you must define `rho`).

```
z1 <- rnorm(1000)
z2 <- rnorm(1000,rho*z1,1-rho^2)
```

Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.:

- 6.37, 6.42, 6.46, 6.57
- 6.68