STAT 512 hw 2

1. Let U_1 and U_2 be independent Uniform$(0, 1)$ random variables and let $Y = U_1U_2$.

   (a) Write down the joint pdf of U_1 and U_2.

   $$f_{U_1,U_2}(u_1, u_2) = 1 \cdot \mathbf{1}(0 < u_1 < 1, 0 < u_2 < 1).$$

   (b) Find the cdf of Y by obtaining an expression for $F_Y(y) = P(Y \leq y) = P(U_1U_2 \leq y)$ for all y.

   We have
   $$F_Y(y) = F_Y(y) = P(Y \leq y) = P(U_1U_2 \leq y) = P(U_2 \leq y/U_1).$$
   For $y \in (0, 1)$, this probability is given by
   $$\begin{aligned}\int_0^y \int_0^1 1 \cdot du_2 du_1 + \int_y^1 \int_0^{y/u_1} 1 \cdot du_2 du_1 &= y + \int_y^1 y/u_1 du_1 \\ &= y + [y \log u_1]\Big|_y^1 \\ &= y + [y \log 1 - y \log y] \\ &= y - y \log y.\end{aligned}$$

   Draw a picture to understand where these integrals come from! So we have
   $$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ y - y \log y, & 0 < y < 1 \\ 1, & y \geq 1. \end{cases}$$

   (c) Find the pdf of Y by taking the derivative of $F_Y(y)$ with respect to y.

   For $y \in (0, 1)$ we have
   $$\frac{d}{dy} F_Y(y) = 1 - (y/y + 1 \cdot \log y) = -\log y.$$
   so the pdf of Y is given by
   $$f_Y(y) = (-\log y)\mathbf{1}(0 < y < 1).$$

   (d) Let $X = U_2$ and find the joint pdf of the rv pair (X, Y) using the bivariate transformation method. Be careful when defining the joint support of (X, Y).

We have

$$\begin{aligned} y &= u_1 u_2 =: g_1(u_1,u_2) \\ x &= u_2 =: g_2(u_1,u_2) \end{aligned} \iff \begin{aligned} u_1 &= y/x =: g_1^{-1}(x,y) \\ u_2 &= x =: g_2^{-1}(x,y) \end{aligned}$$

with Jacobian

$$J(x,y) = \begin{vmatrix} \frac{d}{dx} y/x & \frac{d}{dy} y/x \\ \frac{d}{dx} x & \frac{d}{dy} x \end{vmatrix} = \begin{vmatrix} -(y/x^2) & 1/x \\ 1 & 0 \end{vmatrix} = -1/x.$$

Note that the support of (X,Y) is the set of values $\{x, y : 0 < y < x < 1\}$. So the joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = 1 \cdot |-1/x| \mathbf{1}(0 < y < x < 1) = \frac{1}{x} \mathbf{1}(0 < y < x < 1).$$

(e) Integrate the joint pdf of (X,Y) over X in order to get the pdf of Y.

For $y \in (0,1)$ we have

$$f_Y(y) = \int_y^1 \frac{1}{x} dx = \log x \Big|_y^1 = \log 1 - \log y = -\log y,$$

so

$$f_Y(y) = (-\log y)\mathbf{1}(0 < y < 1).$$

2. Let X_1 and X_2 be independent $\text{Exponential}(1)$ rvs.

(a) Find the joint density of $U_1 = X_1$ and $U_2 = \log(X_1 + X_2) - \log(X_1)$.

The joint density of X_1 and X_2 is given by

$$f_{X_1,X_2}(x_1,x_2) = e^{-(x_1+x_2)} \mathbf{1}(x_1 > 0, x_2 > 0),$$

and we have

$$\begin{aligned} u_1 &= x_1 =: g_1(x_1,x_2) \\ u_2 &= \log(x_1+x_2) - \log(x_1) =: g_2(x_1,x_2) \end{aligned} \iff \begin{aligned} x_1 &= u_1 =: g_1^{-1}(u_1,u_2) \\ x_2 &= u_1 e^{u_2} - u_1 =: g_2^{-1}(u_1,u_2) \end{aligned}$$

with Jacobian

$$J(u_1,u_2) = \begin{vmatrix} \frac{d}{du_1} u_1 & \frac{d}{du_2} u_1 \\ \frac{d}{du_1} u_1 e^{u_2} - u_1 & \frac{d}{du_2} u_1 e^{u_2} - u_1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ e^{u_2} - 1 & u_1 e^{u_2} \end{vmatrix} = u_1 e^{u_2}.$$

Note that the support of (U_1,U_2) is $(0,\infty) \times (0,\infty)$. The joint pdf of U_1 and U_2 is given by

$$\begin{aligned} f_{U_1,U_2}(u_1,u_2) &= e^{-(u_1 + u_1 e^{u_2} - u_1)} \cdot |u_1 e^{u_2}| \cdot \mathbf{1}(u_1 > 0, u_2 > 0) \\ &= u_1 e^{u_2} e^{-u_1 e^{u_2}} \mathbf{1}(u_1 > 0, u_2 > 0). \end{aligned}$$

(b) Show that $U_2 \sim \text{Exponential}(1)$.

For $u_2 \in (0, \infty)$, the marginal pdf of U_2 is given by

$$f_{U_2}(u_2) = \int_0^\infty u_1 e^{u_2} e^{-u_1 e^{u_2}} du_1 = e^{-u_2},$$

since the integral is equivalent to the expected value of an $\text{Exponential}(e^{-u_2})$ random variable. So we have

$$f_{U_2}(u_2) = e^{-u_2}\mathbf{1}(u_2 > 0),$$

which is the pdf of the $\text{Exponential}(1)$ distribution.

(c) Tell whether U_1 and U_2 are independent.

They are not independent because the joint density cannot be factored into the product of a function of u_1 and a function of u_2.

3. Let $G_1 \sim \text{Gamma}(\alpha_1, \beta)$ and $G_2 \sim \text{Gamma}(\alpha_2, \beta)$ and let G_1 and G_2 be independent. Define $B_1 = G_1/(G_1 + G_2)$ and $B_2 = G_1 + G_2$.

(a) Find the joint pdf of (B_1, B_2).

The joint pdf of (G_1, G_2) is given by

$$f_{G_1,G_2}(g_1, g_2) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1}\beta^{\alpha_2}} g_1^{\alpha_1-1} g_2^{\alpha_2-1} e^{-g_1/\beta} e^{-g_2/\beta}\mathbf{1}(g_1 > 0, g_2 > 0),$$

and we have

$$\begin{array}{l} b_1 = g_1/(g_1+g_2) \\ b_2 = g_1 + g_2 \end{array} \iff \begin{array}{l} g_1 = b_1 b_2 \\ g_2 = b_2(1 - b_1) \end{array}$$

with Jacobian

$$J(b_1, b_2) = \left| \begin{array}{cc} \frac{d}{db_1} b_1 b_2 & \frac{d}{db_2} b_1 b_2 \\ \frac{d}{db_1} b_2(1-b_1) & \frac{d}{db_2} b_2(1-b_1) \end{array} \right| = \left| \begin{array}{cc} b_2 & b_1 \\ -b_2 & 1-b_1 \end{array} \right| = b_2(1-b_1) - b_1 b_2 = b_2.$$

Note that the support of (B_1, B_2) is $(0, 1), (0, \infty)$, over which the joint pdf of B_1 and B_2 is given by

$$\begin{aligned} f_{B_1,B_2}(b_1, b_2) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1}\beta^{\alpha_2}} (b_1 b_2^{\alpha_1-1})[b_2(1-b_2)]^{\alpha_2-1} \exp\left[-\frac{b_1 b_2 + b_2(1-b_1)}{\beta}\right] \cdot |b_2| \\ &= \frac{1}{\beta^{\alpha_1+\alpha_2}} b_2^{\alpha_1+\alpha_2-1} e^{-b_2/\beta} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} b_1^{\alpha_1-1}(1-b_1)^{\alpha_2-1}. \end{aligned}$$

(b) Check whether B_1 and B_2 are independent.

The rvs B_1 and B_2 are independent since their joint pdf can be factored into the product of a function of b_2 and a function of b_1, as seen above.

(c) Give the marginal pdf of B_1 and identify its distribution.

Multiplying and dividing the expression for the joint pdf of B_1 and B_2 by $\Gamma(\alpha_1+\alpha_2)$, we see that it becomes the product of two pdfs:

$$f_{B_1,B_2}(b_1,b_2) = \underbrace{\frac{1}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} b_2^{\alpha_1+\alpha_2-1} e^{-b_2/\beta}}_{\text{pdf of Gamma}(\alpha_1+\alpha_2,\beta)} \underbrace{\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} b_1^{\alpha_1-1}(1-b_1)^{\alpha_2-1}}_{\text{pdf of Beta}(\alpha_1,\alpha_2)}.$$

Were we to integrate with respect to b_1 from 0 to 1, we would obtain the Beta(α_1, α_2) pdf; that is

$$f_{B_1}(b_1) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} b_1^{\alpha_1-1}(1-b_1)^{\alpha_2-1}\mathbf{1}(0 < b_1 < 1).$$

(d) Give the marginal pdf of B_2 and identify its distribution.

Were we to integrate the joint pdf with respect to b_2 from 0 to ∞, we would obtain the Gamma$(\alpha_1+\alpha_2, \beta)$ pdf; that is

$$f_{B_2}(b_2) = \frac{1}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} b_2^{\alpha_1+\alpha_2-1} e^{-b_2/\beta}\mathbf{1}(b_2 > 0).$$

4. Let $X_1, \dots, X_N$ be independent rvs and $n_1, \dots, n_N$ be positive integers such that $X_i \sim$ Binomial(n_i, p) for $i = 1, \dots, N$. Give the pmf of $Y = X_1 + \dots + X_N$.

The mgf of Y is given by

$$M_Y(t) = \prod_{i=1}^{N} [pe^t + (1-p)]^{n_i} = [pe^t + (1-p)]^{n_1+\dots+n_N},$$

so Y has the Binomial$(n_1 + \dots + n_N, p)$ distribution.

5. Let $X_1, \dots, X_N$ be independent rvs and $\lambda_1, \dots, \lambda_n$ be positive real numbers such that $X_i \sim$ Poisson(λ_i) for $i = 1, \dots, n$. Give the pmf of $Y = X_1 + \dots + X_n$.

The mgf of Y is given by

$$M_Y(t) = \prod_{i=1}^{N} e^{\lambda_i(e^t-1)} = e^{(\lambda_1+\cdots+\lambda_n)(e^t-1)},$$

so Y has the $\text{Poisson}(\lambda_1 + \cdots + \lambda_n)$ distribution.

6. Let $X_1, \ldots, X_{25}$ be independent $\text{Normal}(\mu = 1, \sigma^2 = 5)$ rvs. Find the distributions of the following:
   (a) $Y_{25} = X_1 + \cdots + X_{25}$

   The mgf of Y is given by

   $$M_Y(t) = [e^{t+5t^2/2}]^{25} = e^{25t+125^2t^2/2},$$

   so $Y \sim \text{Normal}(\mu = 25, \sigma^2 = 125)$.

   (b) $\bar{X}_{25} = \frac{1}{25}(X_1 + \cdots + X_{25})$

   The mgf of $\bar{X}$ is given by

   $$M_{\bar{X}_n}(t) = M_{(X_1+\cdots+X_n)/25}(t) = M_{X_1+\cdots+X_n}(t/25) = [e^{t/25+5(t/25)^2/2}]^{25} = e^t + (1/5)t/2,$$

   so $\bar{X}_n \sim \text{Normal}(\mu = 1, \sigma^2 = 1/5)$.

7. Let Y_1 and Y_2 be independent rvs such that $Y_1 \sim \text{Normal}(\mu_1, \sigma_1^2)$ and $Y_2 \sim \text{Normal}(\mu_2, \sigma_2^2)$ and let $a_1, a_2 \in \mathbb{R}$. Find the distribution of the random variable $a_1Y_1 + a_2Y_2$ using mgfs.

We have

$$\begin{aligned} M_{a_1Y_1+a_2Y_2} &= M_{a_1Y_1}(t)M_{a_2Y_2}(t) \\ &= M_{Y_1}(a_1t)M_{Y_2}(a_2t) \\ &= e^{\mu_1(a_1)t+\sigma_1^2(a_1t)^2/2}e^{\mu_2(a_2)t+\sigma_2^2(a_2t)^2/2} \\ &= e^{(a_1\mu_1+a_2\mu_2)t+(a_1^2\sigma_1^2+a_2^2\sigma_2^2)t^2/2}, \end{aligned}$$

so that $a_1Y_1 + a_2Y_2 \sim \text{Normal}(a_1\mu_1 + a_2\mu_2, a_1^2\sigma_1^2 + a_2^2\sigma_2^2)$.

8. Suppose (Z_1, Z_2) are standard bivariate Normal rvs with correlation ρ, with joint pdf

$$f_{(Z_1,Z_2)}(z_1, z_2) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{1-\rho^2}\right].$$

Choose a value of ρ between 0.5 and 0.9 and use R to generate 1,000 realizations of (Z_1, Z_2). Then transform these into realizations of $U_1 = Z_1 + Z_2$ and $U_2 = Z_1 - Z_2$. Then make two scatter plots: one of the Z_2 values against the Z_1 values and one of the U_2 values against the U_1 values. Turn in your code and these two plots, and say whether you think U_1 and U_2 are independent.

The following code will generate the (Z_1, Z_2) realizations (you must define `rho`).

```
z1 <- rnorm(1000)
z2 <- rnorm(1000,rho*z1,1-rho^2)
```

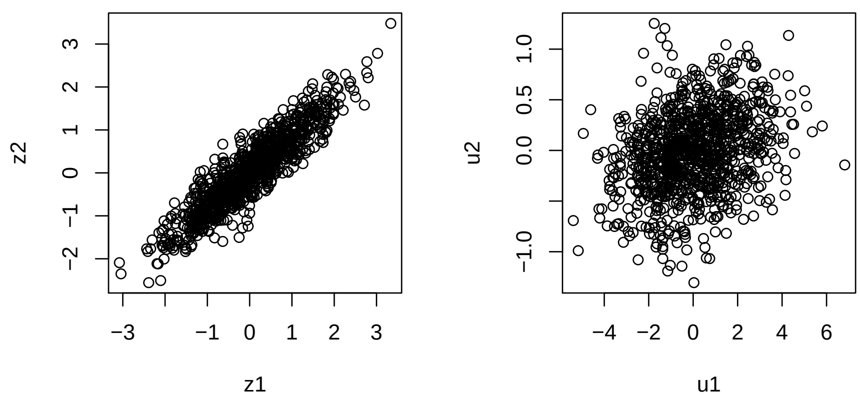
The R code

```
rho <- .8
z1 <- rnorm(1000)
z2 <- rnorm(1000,rho*z1,1-rho^2)

u1 <- z1 + z2
u2 <- z1 - z2

par(mfrow=c(1,2))
plot(z2~z1)
plot(u2~u1)
```

produces the plots



It appears that U_1 and U_2 are independent (they are indeed; see class notes for this example).

Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.:

- 6.37, 6.42, 6.46, 6.57

- 6.68