STAT 512 su 2021 hw 3

1. Let $X_1, \dots, X_{25} \overset{\text{ind}}{\sim} \text{Normal}(\mu = 3, \sigma^2 = 4)$.
   (a) Give the mgf of X_1.
   (b) Give the mgf of $\bar{X}_{25} = (1/25)(X_1 + \cdots + X_{25})$.
   (c) Give $P(X_1 < 2)$.
   (d) Give $P(\bar{X}_{25} < 2)$.
   (e) Give $P(|X_1 - 3| > 1)$.
   (f) Give $P(|\bar{X}_{25} - 3| > 1)$.
   (g) Identify the distribution of $5(\bar{X}_{25} - 3)/2$.
   (h) Give $P([5(\bar{X}_{25} - 3)/2]^2 > 3.841459)$. *Hint: Use the result from Question 2.*
   (i) Now let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu = 3, \sigma^2 = 4)$ for $n \geq 1$ and set $\bar{X}_n = (1/n)(X_1 + \cdots + X_n)$.
      i. Consider the probability $P(|\bar{X}_n - 3| > \varepsilon)$, for some small $\varepsilon > 0$. Is this an increasing or a decreasing function of n?
      ii. What does your answer say about the quality of $\bar{X}_n$ as an estimator of the mean?

2. Let $Z \sim \text{Normal}(0, 1)$. Show that Z^2 has the χ^2_1 distribution (chi-squared with degrees of freedom 1), i.e. show that the pdf of $Y = Z^2$ is

$$f_Y(y) = \frac{1}{\Gamma(1/2)2^{1/2}} y^{\frac{1}{2}-1} e^{-y/2} \mathbf{1}(y > 0).$$

   Hint: $\Gamma(1/2) = \sqrt{\pi}$.

3. Let $W \sim \text{Chi-squared}(\nu)$, so that the pdf of W is given by

$$f_W(w) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}} w^{\nu/2-1} e^{-w/2} \mathbf{1}(w > 0),$$

   where $\nu > 0$ is the degrees of freedom.
   (a) Give α and β such the $\text{Gamma}(\alpha, \beta)$ and $\text{Chi-squared}(\nu)$ distributions are the same.
   (b) Give $\mathbb{E}W$ in terms of the degrees of freedom parameter ν.
   (c) Give $\text{Var}\, W$ in terms of the degrees of freedom parameter ν.

4. Let $Y_1, \dots, Y_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$ distribution, $p \in (0, 1)$. Let $\hat{p}_n = \bar{Y}_n = n^{-1}(Y_1 + \cdots + Y_n)$.
   (a) Show that

$$\frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y}_n)^2 = \frac{n}{n-1} \hat{p}_n (1 - \hat{p}_n).$$

   (b) Find $\mathbb{E}\left[n(n-1)^{-1} \hat{p}_n (1 - \hat{p}_n)\right]$.

5. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim}$ Poisson(λ) distribution. Find the expected value of $\hat{\lambda} = n^{-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$. *Hint:* $\mathbb{E}\hat{\lambda} \neq \lambda$.

6. Let $X_1, \dots, X_n$ be a random sample from the Weibull(k, λ) distribution, which has pdf

$$f(x) = \begin{cases} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

for $k, \lambda > 0$.

   (a) Find the cdf F_X of the Weibull(k, λ) distribution. *Hint: Set up the integral* $F_X(x) = \int_0^x f_X(t)dt$ *and do the change of variable* $u = (t/\lambda)^k$.

   (b) Find the pdf of $X_{(1)}$.

   (c) Show that $X_{(1)}$ has the Weibull$(k, \lambda n^{-1/k})$ distribution.

7. Let $U_1, \dots, U_n$ be a random sample from the Uniform$(0, \theta)$ distribution.

   (a) Find the joint density of order statistics $U_{(1)}$ and $U_{(n)}$.

   (b) Find the joint density of the random variables $R = U_{(1)}/U_{(n)}$ and $M = U_{(n)}$.

   (c) State whether R and M are independent.

   (d) Give the marginal pdf of R and identify the distribution.

8. Let $X_1, \dots, X_n$ be a random sample from the Uniform$(0, 1)$ distribution, where n is an odd number. Show that

   (a) The expected value of the median is $1/2$.

   (b) The variance of the median is $\frac{1}{4}\frac{1}{n+2}$.

9. Use R to run the following simulation. Choose a sample size $n \leq 20$ and draw $1{,}000$ samples of size n from the Uniform$(0, 1)$ distribution. In so doing:

   (a) Choose a value of k, $1 < k < n$, and from each of the $1{,}000$ samples, save the kth order statistic. Make a histogram of the $1{,}000$ values of the kth order statistic and overlay the pdf of the sampling distribution of the kth order statistic (you must figure out and input the shape parameters of the beta distribution). Use the following code to get started:

```
S <- 1000 # number of random samples to generate
Yk <- numeric(S) # create empty vector in which to store values

for(s in 1:S) # run a loop of length S
{
    Y <- runif(n)
    Yk[s] <- sort(Y)[k] # get kth order statistic
}

hist(Yk,freq=FALSE,xlim=c(0,1))
y.seq <- seq(0,1,length=100)
lines(dbeta(y.seq,shape1=???,shape2=???)~y.seq,col="blue",lwd=2)
```

   (b) Do the same thing for the nth order statistic and the 1st order statistic.

Turn in all three histograms with densities overlaid and all your code.