STAT 512 hw 4

1. Let $X \sim \text{Normal}(\mu, 1)$.
   (a) Give an interval (L, U), where U and L are based on X, such that $P(L < \mu < U) = 0.99$.
   (b) Give an upper bound U based on X such that $P(\mu < U) = 0.99$.
   (c) Give a lower bound L based on X such that $P(L < \mu) = 0.99$.

2. Let $X \sim \text{Normal}(0, \sigma^2)$.
   (a) Find the distribution of X^2/σ^2. *Hint: It is a pivot quantity.*
   (b) Give an interval (L, U), where U and L are based on X, such that $P(L < \sigma^2 < U) = 0.95$.
   (c) Give an upper bound U based on X such that $P(\sigma^2 < U) = 0.95$.
   (d) Give a lower bound L based on X such that $P(L < \sigma^2) = 0.95$.

3. The following data from [1] are carapace lengths (mm) of lobsters caught in some region:

   78 66 65 63 60 60 58 56 52 50

   Assume that the lengths of lobster carapaces in the region have a Normal distribution.
   (a) Give a 95% confidence interval for the mean carapace length of lobsters in the region.
   (b) Give a 95% confidence interval for the variance of carapace lengths of lobsters in the region.

4. Load the `PlantGrowth` data set in R by executing the command `data(PlantGrowth)`. Assume that the weights in all the groups come from a Normal distribution.
   (a) Give a 95% confidence interval for σ_2^2/σ_1^2, where σ_1^2 is the variance of weights in the `ctrl` group and σ_2^2 is the variance of the weights in the `trt1` group.
   (b) If $X_1, \dots, X_{n_1} \overset{\text{ind}}{\sim} \text{Normal}(\mu_1, \sigma_1^2)$ and $Y_1, \dots, Y_{n_2} \overset{\text{ind}}{\sim} \text{Normal}(\mu_2, \sigma_2^2)$ and if $\sigma_1^2 = \sigma_2^2$, then we have the pivot quantity result

   $$\frac{\bar{X}_{n_1} - \bar{Y}_{n_2} - (\mu_1 - \mu_2)}{\sqrt{S^2_{\text{pooled}}(1/n_1 + 1/n_2)}} \sim t_{n_1+n_2-2},$$

   where

   $$S^2_{\text{pooled}} = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}.$$

   Assuming that the variances of the plant growths in the `ctrl` and the `trt1` groups are equal, construct a 95% confidence interval for $\mu_1 - \mu_2$, where μ_1 is the mean of weights in the `ctrl` group and μ_2 is the mean of the weights in the `trt1` group.

5. Let Z_1, Z_2, Z_3, and Z_4 be independent random variables having the $\text{Normal}(0, 1)$ distribution and let $\bar{Z} = (Z_1 + Z_2 + Z_3 + Z_4)/4$. Give the distributions of the following quantities.
   (a) $\bar{Z}$
   (b) $Z_1^2 + Z_2^2 + Z_3^2$

(c) $Z_1/\sqrt{(Z_2^2 + Z_3^2)/2}$

(d) $2\bar{Z}[(1/3)\sum_{i=1}^{4}(Z_i - \bar{Z}_4)^2]^{-1/2}$

(e) $\sum_{i=1}^{4}(Z_i - \bar{Z}_4)^2$

(f) $(Z_1^2 + Z_2^2)/(Z_3^2 + Z_4^2)$

6. Let Z and W be independent random variables such that $Z \sim \text{Normal}(0, 1)$ and $W \sim \chi^2_\nu$, for some $\nu > 0$. Recall that the random variable

$$T = \frac{Z}{\sqrt{W/\nu}} \sim t_\nu.$$

   (a) Find the mean of a random variable having the t_ν distribution.

   (b) Show that $\mathbb{E}W^{-1} = 1/(\nu - 2)$, assuming $\nu > 2$.

   (c) Find the variance of a random variable having the t_ν distribution.

7. Two javelin throwers will compete. Suppose the distances thrown by the first follow the $\text{Normal}(70, 25)$ distribution (units in meters) and those of the second follow the $\text{Normal}(68, 36)$ distribution. Give the probability that the second javelin thrower throws further than the first, assuming that their throws are independent.

References

[1] William B Jeffries, Chang Man Yang, and Harold K Voris. Diversity and distribution of the pedunculate barnacle octolasmis gray, 1825 epizoic on the scyllarid lobster, thenus orientalis (lund, 1793). *Crustaceana*, 46(3):300–308, 1984.