STAT 512 hw 8

1. Let $X_1, X_2, X_3 \overset{\text{iid}}{\sim} \text{Exponential}(\lambda)$.
   (a) Show that $T(X_1, X_2, X_3) = X_1 + X_2 + X_3$ is a sufficient statistic for λ.
   (b) Find the MVUE for λ.
   (c) Show that $X_{(1)}$ is not a sufficient statistic for λ.
   (d) Let $\tilde{\lambda} = 3X_{(1)}$ and find $\mathbb{E}\tilde{\lambda}$. Give an argument for why $\tilde{\lambda}$ is not the best estimator of λ.

2. Let $X_1, \ldots, X_n$ be a random sample from the distribution with pdf given by

$$f_X(x; \beta) = \frac{\beta}{x^{\beta+1}} \mathbf{1}(x \geq 1).$$

   (a) Show that $T = \sum_{i=1}^n \log X_i$ is a sufficient statistic for β. *Hint:* Use

$$\prod_{i=1}^n \frac{1}{x_i} = \exp\left[\log \prod_{i=1}^n \frac{1}{x_i}\right] = \exp\left[-\sum_{i=1}^n \log x_i\right].$$

   (b) Find the pdf of $Y = \log X$, where $X \sim f_X(x; \beta)$.
   (c) Find the distribution of T. *Hint: Identify the distribution of Y and use mgfs.*
   (d) Find $\mathbb{E}[1/T]$.
   (e) Use your answer to the previous part to propose an unbiased estimator of β based on T.
   (f) Argue that your answer to part (c) is the MVUE of β.

3. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Beta}(\alpha, \beta)$. Show that $(\prod_{i=1}^n X_i, \prod_{i=1}^n (1 - X_i))$ is a sufficient statistic for (α, β).

4. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(\mu - \theta, \mu + \theta)$.
   (a) Give the pdf of the $\text{Uniform}(\mu - \theta, \mu + \theta)$ distribution.
   (b) Show that $(X_{(1)}, X_{(n)})$ is a sufficient statistic for (μ, θ) using the factorization theorem.
   Hint: $\prod_{i=1}^n \mathbf{1}(\mu - \theta < x_i < \mu + \theta) = \mathbf{1}(\mu - \theta < x_1, ..., x_n < \mu + \theta)$.
   (c) Give the cdf of the $\text{Uniform}(\mu - \theta, \mu + \theta)$ distribution.
   (d) Consider the order statistic $X_{(1)}$:
      i. Find the pdf of $X_{(1)}$.
      ii. Find the pdf of

$$Y_{(1)} = \frac{X_{(1)} - (\mu - \theta)}{2\theta}$$

      and identify its distribution.
      iii. Find $\mathbb{E}X_{(1)}$. *Hint: Use the fact that $X_{(1)} = 2\theta Y_{(1)} + (\mu - \theta)$.*
   (e) Consider the order statistic $X_{(n)}$:

i. Find the pdf of $X_{(n)}$.

ii. Find the pdf of

$$Y_{(n)} = \frac{X_{(n)} - (\mu - \theta)}{2\theta}$$

and identify its distribution.

iii. Find $\mathbb{E}X_{(n)}$. *Hint: Use the fact that* $X_{(n)} = 2\theta Y_{(n)} + (\mu - \theta)$.

(f) Consider the estimators of θ and μ given by

$$\hat{\theta} = \frac{X_{(n)} - X_{(1)}}{2} \quad \text{and} \quad \hat{\mu} = \frac{X_{(1)} + X_{(n)}}{2}.$$

i. Find $\mathbb{E}\hat{\theta}$ and state whether the estimator is biased or unbiased.

ii. Find $\mathbb{E}\hat{\mu}$ and state whether the estimator is biased or unbiased.

iii. Propose an unbiased estimator of θ.

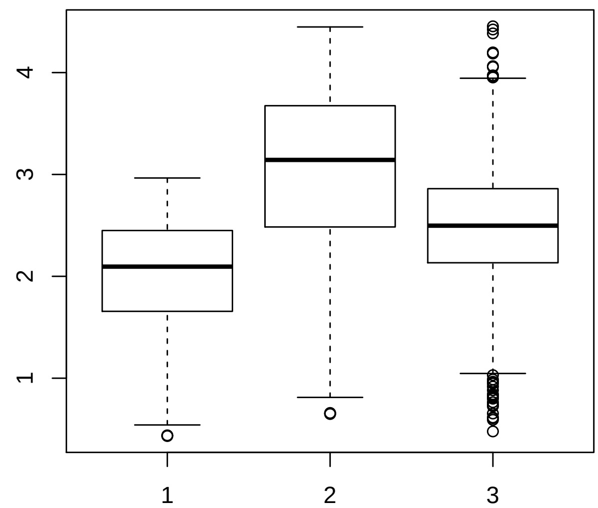
iv. Argue whether there might exist another unbiased estimator of θ with smaller variance.

(g) The following code stores values in the vectors `theta.hat`, `mu.hat`, and `theta.hat.unbiased`.

```
mu <- 2.5
theta <- 3
n <- 5
S <- 1000

theta.hat <- numeric(S)
theta.hat.unbiased <- numeric(S)
mu.hat <- numeric(S)
for( s in 1:S ){
  X <- runif(n,mu - theta, mu + theta)
  X1 <- min(X)
  Xn <- max(X)
  theta.hat[s] <- (Xn - X1)/2
  mu.hat[s] <- (X1 + Xn)/2
  theta.hat.unbiased[s] <- (n+1)/(n-1) * (Xn - X1)/2
}
```

The figure shows boxplots of the values of `theta.hat`, `mu.hat`, and `theta.hat.unbiased` from the simulation. Study the code and identify which boxplot corresponds to `theta.hat`, `mu.hat`, and `theta.hat.unbiased`.



Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.:

- 9.50, 9.59
- 9.74, 9.75, 9.78