STAT 512 hw 8

1. Let $X_1, X_2, X_3 \overset{\text{iid}}{\sim} \text{Exponential}(\lambda)$.

   (a) Show that $T(X_1, X_2, X_3) = X_1 + X_2 + X_3$ is a sufficient statistic for λ.

   We have

$$
\begin{aligned}
f_{X_1,X_2,X_3}(x_1,x_2,x_3) &= \frac{1}{\lambda}e^{-x_1/\lambda}\mathbf{1}(x_1>0)\cdot\frac{1}{\lambda}e^{-x_2/\lambda}\mathbf{1}(x_2>0)\cdot\frac{1}{\lambda}e^{-x_3/\lambda}\mathbf{1}(x_3>0)\\
&= \underbrace{\left(\frac{1}{\lambda}\right)^3 e^{-(x_1+x_2+x_3)/\lambda}}_{g(x_1+x_2+x_3;\lambda)}\cdot\underbrace{\mathbf{1}(x_1>0)\mathbf{1}(x_2>0)\mathbf{1}(x_3>0)}_{h(x_1,x_2,x_3)}.
\end{aligned}
$$

   Since we can factor the joint density $f_{X_1,X_2,X_3}(x_1,x_2,x_3)$ of X_1, X_2, X_3 into the product of a function of $x_1+x_2+x_3$ and the parameter λ and a function of x_1, x_2, x_3, we can say, by the factorization theorem, that $X_1+X_2+X_3$ is a sufficient statistic for λ.

   (b) Find the MVUE for λ.

   To find the MVUE, we find a function of the statistic $X_1+X_2+X_3$, which is a sufficient statistic for λ, which is unbiased for λ. The MVUE is

$$
\hat{\lambda} = \frac{X_1+X_2+X_3}{3},
$$

   since $\mathbb{E}\hat{\lambda} = \lambda$ and since $\hat{\lambda}$ is a function of $X_1+X_2+X_3$.

   (c) Show that $X_{(1)}$ is not a sufficient statistic for λ.

   We can use the factorization theorem or we can check whether λ cancels out of the ratio of the joint density of X_1, X_2, X_3 and the pdf of $X_{(1)}$. For the factorization approach, we have, for $x_1, x_2, x_3 > 0$,

$$
\begin{aligned}
f_{X_1,X_2,X_3}(x_1,x_2,x_3) &= \left(\frac{1}{\lambda}\right)^3 e^{-(x_1+x_2+x_3)/\lambda}\\
&= \left(\frac{1}{\lambda}\right)^3 e^{-(x_{(1)}+x_{(2)}+x_{(3)})/\lambda}, \quad x_{(1)}<x_{(2)}<x_{(3)}\\
&= \left(\frac{1}{\lambda}\right)^3 e^{-x_{(1)}/\lambda}e^{-(x_{(2)}+x_{(3)})/\lambda}.
\end{aligned}
$$

   At this point we see that there is no way to factorize $f_{X_1,X_2,X_3}(x_1,x_2,x_3)$ as the product of a function of $x_{(1)}$ and λ and a function of x_1, x_2, x_3. For the second approach we must find the pdf of $X_{(1)}$, which is

$$
f_{X_{(1)}}(x_{(1)}) = 3\left[1-(1-e^{-x_{(1)}/\lambda})\right]^2\frac{1}{\lambda}e^{-x_{(1)}/\lambda} = \frac{1}{\lambda/3}e^{-x_{(1)}/(\lambda/3)}.
$$

Now we check whether λ cancels out of the ratio

$$\frac{f_{X_1,X_2,X_3}(x_1,x_2,x_3)}{f_{X_{(1)}}(x_{(1)})} = \frac{\left(\frac{1}{\lambda}\right)^3 e^{-(x_{(1)}+x_{(2)}+x_{(3)})/\lambda}}{\frac{1}{\lambda/3}e^{-x_{(1)}/(\lambda/3)}}.$$

We see that λ does not cancel out of the above ratio, so $X_{(1)}$ is not a sufficient statistic for λ.

(d) Let $\tilde{\lambda} = 3X_{(1)}$ and find $\mathbb{E}\tilde{\lambda}$. Give an argument for why $\tilde{\lambda}$ is not the best estimator of λ.

We determined in the solution to the previous part that $X_{(1)} \sim \text{Exponential}(\lambda/3)$, so that $\mathbb{E}X_{(1)} = \lambda/3$. Therefore $\mathbb{E}\tilde{\lambda} = 3(\lambda/3) = \lambda$, so that $\tilde{\lambda}$ is unbiased. However, $\tilde{\lambda}$ is not the MVUE because it is not a function of a sufficient statistic for λ.

2. Let $X_1, \dots, X_n$ be a random sample from the distribution with pdf given by

$$f_X(x;\beta) = \frac{\beta}{x^{\beta+1}}\mathbf{1}(x \geq 1).$$

(a) Show that $T = \sum_{i=1}^n \log X_i$ is a sufficient statistic for β. *Hint:* Use

$$\prod_{i=1}^n \frac{1}{x_i} = \exp\left[\log \prod_{i=1}^n \frac{1}{x_i}\right] = \exp\left[-\sum_{i=1}^n \log x_i\right].$$

We have

$$\begin{aligned} f_{X_1,\dots,X_n}(x_1,\dots,x_n) &= \prod_{i=1}^n \frac{\beta}{x_i^{\beta+1}}\mathbf{1}(x_i \geq 1) \\ &= \beta^n \left[\prod_{i=1}^n \frac{1}{x_i}\right]^{\beta+1} \prod_{i=1}^n \mathbf{1}(x_i \geq 1) \\ &= \underbrace{\beta^n \left[\exp\left(-\sum_{i=1}^n \log x_i\right)\right]^{\beta+1}}_{g(\sum_{i=1}^n \log x_i;\beta)} \underbrace{\prod_{i=1}^n \mathbf{1}(x_i \geq 1)}_{h(x_1,\dots,x_n)}. \end{aligned}$$

By the factorization theorem, $T = \sum_{i=1}^n \log X_i$ is a sufficient statistic for β.

(b) Find the pdf of $Y = \log X$, where $X \sim f_X(x;\beta)$.

We have

$$y = \log x =: g(x) \iff x = e^y =: g^{-1}(y), \quad \frac{d}{dy} g^{-1}(y) = e^y$$

and $X \in [1, \infty) \implies Y \in [0, \infty)$. So by the transformation method the pdf of Y is given by

$$f_Y(y) = \frac{\beta}{e^{y(\beta+1)}} e^y \mathbf{1}(y \geq 0) = \beta e^{-y\beta} \mathbf{1}(y \geq 0),$$

so that $Y \sim \text{Exponential}(1/\beta)$.

(c) Find the distribution of T. *Hint: Identify the distribution of Y and use mgfs.*

The moment generating function of T is given by

$$M_T(t) = M_{\sum_{i=1}^n \log X_i}(t) = M_{\sum_{i=1}^n Y_i}(t) = [M_Y(t)]^n = \left[(1 - (1/\beta)t)^{-1}\right]^n = (1 - (1/\beta))^{-n}.$$

We recognize this as the mgf of the $\text{Gamma}(n, 1/\beta)$ distribution, so $T \sim \text{Gamma}(n, 1/\beta)$.

(d) Find $\mathbb{E}[1/T]$.

We have

$$\begin{aligned}
\mathbb{E}[1/T] &= \int_0^\infty \frac{1}{t} \frac{1}{\Gamma(n)(1/\beta)^n} t^{n-1} e^{-t/(1/\beta)} dt \\
&= \frac{\Gamma(n-1)(1/\beta)^{n-1}}{\Gamma(n)(1/\beta)^n} \underbrace{\int_0^\infty \frac{1}{\Gamma(n-1)(1/\beta)^{n-1}} t^{(n-1)-1} e^{-t/(1/\beta)} dt}_{=1, \text{ since integral over Gamma}(n-1, 1/\beta) \text{ pdf}} \\
&= \frac{\beta}{n-1}
\end{aligned}$$

(e) Use your answer to the previous part to propose an unbiased estimator of β based on T.

An unbiased estimator for β is

$$\hat{\beta} = \frac{(n-1)}{T} = \frac{n-1}{\sum_{i=1}^n \log X_i},$$

since $\mathbb{E}\hat{\beta} = \mathbb{E}(n-1)/T = (n-1)\beta/(n-1) = \beta$.

(f) Argue that your answer to part (c) is the MVUE of β.

Since the estimator $\hat{\beta}$ is unbiased for β and since it is a function of a sufficient statistic for β, it is the MVUE, by the Rao-Blackwell theorem.

3. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Beta}(\alpha, \beta)$. Show that $(\prod_{i=1}^n X_i, \prod_{i=1}^n (1 - X_i))$ is a sufficient statistic for (α, β).

The joint density of $X_1, \ldots, X_n$ is given by

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{i=1}^n \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x_i^{\alpha-1}(1-x_i)^{\beta_1}\mathbf{1}(0 < x_i < 1)$$

$$= \underbrace{\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right]^n \left[\prod_{i=1}^n x_i\right]^{\alpha-1} \left[\prod_{i=1}^n (1-x_i)\right]^{\beta_1}}_{g(\prod_{i=1}^n x_i, \prod_{i=1}^n (1-x_i); \alpha, \beta)} \cdot \underbrace{\prod_{i=1}^n \mathbf{1}(0 < x_i < 1)}_{h(x_1,\ldots,x_n)}.$$

Since the joint density $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$ of $X_1, \ldots, X_n$ can be factored into a function of $\prod_{i=1}^n x_i$ and $\prod_{i=1}^n (1 - x_i)$ and α and β and a function of $x_1, \ldots, x_n$, as shown above, we can say by the factorization theorem that $(\prod_{i=1}^n X_i, \prod_{i=1}^n (1 - X_i))$ is a sufficient statistic for (α, β).

4. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(\mu - \theta, \mu + \theta)$.

   (a) Give the pdf of the $\text{Uniform}(\mu - \theta, \mu + \theta)$ distribution.

   The pdf of the $\text{Uniform}(\mu - \theta, \mu + \theta)$ distribution is given by

   $$f_X(x) = \frac{1}{2\theta}\mathbf{1}(\mu - \theta < x < \mu + \theta).$$

   (b) Show that $(X_{(1)}, X_{(n)})$ is a sufficient statistic for (μ, θ) using the factorization theorem. *Hint:* $\prod_{i=1}^n \mathbf{1}(\mu - \theta < x_i < \mu + \theta) = \mathbf{1}(\mu - \theta < x_1, ..., x_n < \mu + \theta)$.

   The joint pdf of $(X_1, \ldots, X_n)$ is given by

   $$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{i=1}^n \frac{1}{2\theta}\mathbf{1}(\mu - \theta < x_i < \mu + \theta)$$

   $$= \left(\frac{1}{2\theta}\right)^n \mathbf{1}(\mu - \theta < x_i, \ldots, x_n < \mu + \theta)$$

   $$= \left(\frac{1}{2\theta}\right)^n \mathbf{1}(\mu - \theta < x_{(1)} < x_{(n)} < \mu + \theta)$$

   $$= \underbrace{\left(\frac{1}{2\theta}\right)^n \mathbf{1}(\mu - \theta < x_{(1)} < x_{(n)} < \mu + \theta)}_{g(x_{(1)}, x_{(n)}; \mu, \theta)} \cdot \underbrace{1}_{h(x_1,\ldots,x_n)}$$

   where $x_{(1)} = \min\{x_1, \ldots, x_n\}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\}$. So by the factorization theorem, we can say that $(X_{(1)}, X_{(n)})$ is a sufficient statistic for (μ, θ).

(c) Give the cdf of the Uniform$(\mu-\theta, \mu+\theta)$ distribution.

The cdf of the Uniform$(\mu-\theta, \mu+\theta)$ distribution is given by

$$F_X(x) = \begin{cases} 0, & x < \mu-\theta \\ \dfrac{x-(\mu-\theta)}{2\theta}, & \mu-\theta \leq x < \mu+\theta \\ 1, & \mu+\theta \leq x \end{cases}$$

(d) Consider the order statistic $X_{(1)}$:

i. Find the pdf of $X_{(1)}$.

We have

$$f_{X_{(1)}}(x) = n\left[1 - \frac{x-(\mu-\theta)}{2\theta}\right]^{n-1}\frac{1}{2\theta}, \quad \mu-\theta < x < \mu+\theta.$$

ii. Find the pdf of

$$Y_{(1)} = \frac{X_{(1)}-(\mu-\theta)}{2\theta}$$

and identify its distribution.

We have

$$y_{(1)} = \frac{x_{(1)}-(\mu-\theta)}{2\theta} =: g(x_{(1)}) \iff x_{(1)} = 2\theta y_{(1)} + (\mu-\theta) =: g^{-1}(y_{(1)})$$

with

$$\frac{d}{dy_{(1)}}g^{-1}(y_{(1)}) = 2\theta,$$

and the support of $Y_{(1)}$ is $(0,1)$. So by the transformation method we have

$$f_{Y_{(1)}}(y) = n(1-y)^{n-1}, \quad 0 < y < 1,$$

which is the pdf of the Beta$(1, n)$ distribution.

iii. Find $\mathbb{E}X_{(1)}$. *Hint: Use the fact that* $X_{(1)} = 2\theta Y_{(1)} + (\mu-\theta)$.

We have

$$\begin{aligned} \mathbb{E}X_{(1)} &= \mathbb{E}2\theta Y_{(1)} + (\mu-\theta) \\ &= 2\theta\left(\frac{1}{n+1}\right) + (\mu-\theta) \\ &= \mu - \left(\frac{n-1}{n+1}\right)\theta, \end{aligned}$$

where we used the fact that the mean of the $\text{Beta}(1, n)$ distribution is $1/(n + 1)$.

(e) Consider the order statistic $X_{(n)}$:

i. Find the pdf of $X_{(n)}$.

$$f_{X_{(n)}}(x) = n \left[\frac{x - (\mu - \theta)}{2\theta} \right]^{n-1} \frac{1}{2\theta}, \quad \mu - \theta < x < \mu + \theta.$$

ii. Find the pdf of

$$Y_{(n)} = \frac{X_{(n)} - (\mu - \theta)}{2\theta}$$

and identify its distribution.

By similar steps to those with which we found the pdf of $Y_{(1)}$, we can show that the pdf of $Y_{(n)}$ is

$$f_{Y_{(n)}}(y) = ny^{n-1}, \quad 0 < y < 1,$$

which is the pdf of the $\text{Beta}(n, 1)$ distribution.

iii. Find $\mathbb{E}X_{(n)}$. *Hint: Use the fact that* $X_{(n)} = 2\theta Y_{(n)} + (\mu - \theta)$.

We have

$$\begin{aligned} \mathbb{E}X_{(n)} &= \mathbb{E}2\theta Y_{(n)} + (\mu - \theta) \\ &= 2\theta \left(\frac{n}{n+1} \right) + (\mu - \theta) \\ &= \mu + \left(\frac{n-1}{n+1} \right) \theta, \end{aligned}$$

where we used the fact that the mean of the $\text{Beta}(n, 1)$ distribution is $n/(n + 1)$.

(f) Consider the estimators of θ and μ given by

$$\hat{\theta} = \frac{X_{(n)} - X_{(1)}}{2} \quad \text{and} \quad \hat{\mu} = \frac{X_{(1)} + X_{(n)}}{2}.$$

i. Find $\mathbb{E}\hat{\theta}$ and state whether the estimator is biased or unbiased.

We have

$$\begin{aligned}\mathbb{E}\hat{\theta} &= \mathbb{E}\left[\frac{X_{(n)} - X_{(1)}}{2}\right] \\ &= \frac{1}{2}\left[\mu + \left(\frac{n-1}{n+1}\right)\theta - \left(\mu - \left(\frac{n-1}{n+1}\right)\theta\right)\right] \\ &= \left(\frac{n-1}{n+1}\right)\theta.\end{aligned}$$

Since $\mathbb{E}\hat{\theta} \neq \theta$, $\hat{\theta}$ is a *biased* estimator of θ.

ii. Find $\mathbb{E}\hat{\mu}$ and state whether the estimator is biased or unbiased.

We have

$$\begin{aligned}\mathbb{E}\hat{\mu} &= \mathbb{E}\left[\frac{X_{(1)} + X_{(n)}}{2}\right] \\ &= \frac{1}{2}\left[\mu - \left(\frac{n-1}{n+1}\right)\theta + \mu + \left(\frac{n-1}{n+1}\right)\theta\right] \\ &= \mu.\end{aligned}$$

Since $\mathbb{E}\hat{\mu} = \mu$, $\hat{\mu}$ is an *unbiased* estimator of μ.

iii. Propose an unbiased estimator of θ.

The estimator

$$\tilde{\theta} = \left(\frac{n+1}{n-1}\right)\frac{X_{(n)} - X_{(1)}}{2}$$

is unbiased for θ, since

$$\mathbb{E}\tilde{\theta} = \left(\frac{n+1}{n-1}\right)\mathbb{E}\left[\frac{X_{(n)} - X_{(1)}}{2}\right] = \left(\frac{n+1}{n-1}\right)\left(\frac{n-1}{n+1}\right)\theta = \theta.$$

iv. Argue whether there might exist another unbiased estimator of θ with smaller variance.

Since $\tilde{\theta}$ is a function of $(X_{(1)}, X_{(n)})$, which is a sufficient statistic for (μ, θ), and since $\tilde{\theta}$ is unbiased, we can say, because of the Rao-Blackwell theorem, that there exists no other unbiased estimator with a smaller variance. This is the one and only precious jewel!

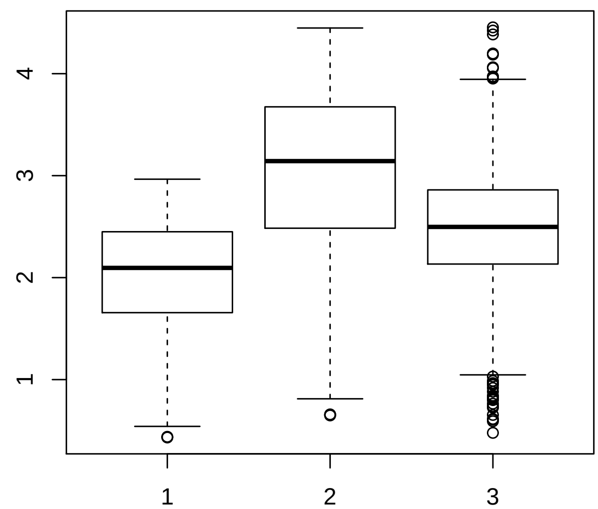
(g) The following code stores values in the vectors `theta.hat`, `mu.hat`, and `theta.hat.unbiased`.

```
mu <- 2.5
theta <- 3
```

```
n <- 5
S <- 1000

theta.hat <- numeric(S)
theta.hat.unbiased <- numeric(S)
mu.hat <- numeric(S)
for( s in 1:S ){
  X <- runif(n,mu - theta, mu + theta)
  X1 <- min(X)
  Xn <- max(X)
  theta.hat[s] <- (Xn - X1)/2
  mu.hat[s] <- (X1 + Xn)/2
  theta.hat.unbiased[s] <- (n+1)/(n-1) * (Xn - X1)/2
}
```

The figure shows boxplots of the values of `theta.hat`, `mu.hat`, and `theta.hat.unbiased` from the simulation. Study the code and identify which boxplot corresponds to `theta.hat`, `mu.hat`, and `theta.hat.unbiased`.



The right-most boxplot is of the values of `mu.hat`, since $\hat{\mu}$ is an unbiased estimator of μ, which is set equal to 2.5, and the boxplot is centered at 2.5.

The boxplot in the middle is of the values of `theta.hat.unbiased`, since this is an unbiased estimator of θ, which is set equal to 3, and the boxplot is centered at 3.

The other boxplot must be of the values of `theta.hat`.

Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.:

- 9.50, 9.59
- 9.74, 9.75, 9.78