STAT 512 su 2021 hw 9

1. Suppose $Y_1, \dots, Y_n \overset{\text{ind}}{\sim} \text{Gamma}(2, \beta)$.
   (a) Write down the likelihood function for β based on $Y_1, \dots, Y_n$.
   (b) Write down the log-likelihood function for β based on $Y_1, \dots, Y_n$.
   (c) Find an expression for the MLE of β.
   (d) Give the MoMs estimator of β.

2. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(\mu - \theta, \mu + \theta)$.
   (a) Give expressions for the MoMs estimators, call them $\bar{\mu}$ and $\bar{\theta}$, of μ and θ.
   Hint: Based on a sample of size $n = 5$ with the values

   ```
   -0.44 0.27 3.08 0.90 0.51
   ```

   the MoMs estimator of θ should have the value `2.062598` *(to double-check your answer).*

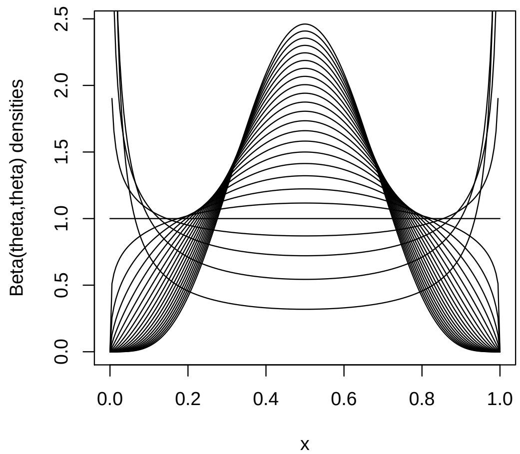
   (b) Now consider the estimators of θ and μ given by

   $$\hat{\theta}_{\text{unbiased}} = \left(\frac{n+1}{n-1}\right) \frac{X_{(n)} - X_{(1)}}{2} \quad \text{and} \quad \hat{\mu} = \frac{X_{(1)} + X_{(n)}}{2},$$

   where $\hat{\theta}_{\text{unbiased}}$ is a bias-corrected, or *de-biased* version of the MLE for θ. Run a simulation in which $1{,}000$ random samples of size $n = 5$ are drawn from the $\text{Uniform}(\mu - \theta, \mu + \theta)$ distribution with $\mu = 2.5$ and $\theta = 3$, and compute the estimators $\bar{\mu}$, $\hat{\mu}$, $\bar{\theta}$, and $\hat{\theta}_{\text{unbiased}}$ on each random sample. Then:
   i. Make side-by-side boxplots of the $1{,}000$ values of $\bar{\mu}$ and $\hat{\mu}$. You can take a photo of your screen to merge into your pdf to upload. Include R code.
   ii. Compare the mean squared error of $\bar{\mu}$ and $\hat{\mu}$ based on the $1{,}000$ simulated data sets and comment on which estimator you think is a better estimator of μ.
   iii. Make side-by-side boxplots of the $1{,}000$ values of $\bar{\theta}$ and $\hat{\theta}_{\text{unbiased}}$. You can take a photo of your screen to merge into your pdf to upload. Include R code.
   iv. Compare the mean squared error of $\bar{\theta}$ and $\hat{\theta}_{\text{unbiased}}$ based on the $1{,}000$ simulated data sets and comment on which estimator you think is a better estimator of θ.

   Hint: Look at the code from hw 7 for setting up the simulation. To make side-by-side boxplots, just use `boxplot(mu.mle,mu.mom)`*, for example.*

3. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Beta}(\theta, \theta)$, $\theta > 0$. The $\text{Beta}(\theta, \theta)$ pdf is plotted below for all $\theta \in \{.2, .4, .6, \dots, 5\}$:



(a) Find the MoMs estimator of θ by setting $m_2' = \mu_2'$ (note that $\mu_1' = 1/2$ for all θ).

(b) Write down the likelihood function $\mathcal{L}(\theta; X_1, \dots, X_n)$.

(c) Argue that $\prod_{i=1}^n X_i(1 - X_i)$ is a sufficient statistic for θ.

(d) Write down the log-likelihood function $\ell(\theta; X_1, \dots, X_n)$.

(e) Use the following code to plot the log-likelihood function based an observed random sample:

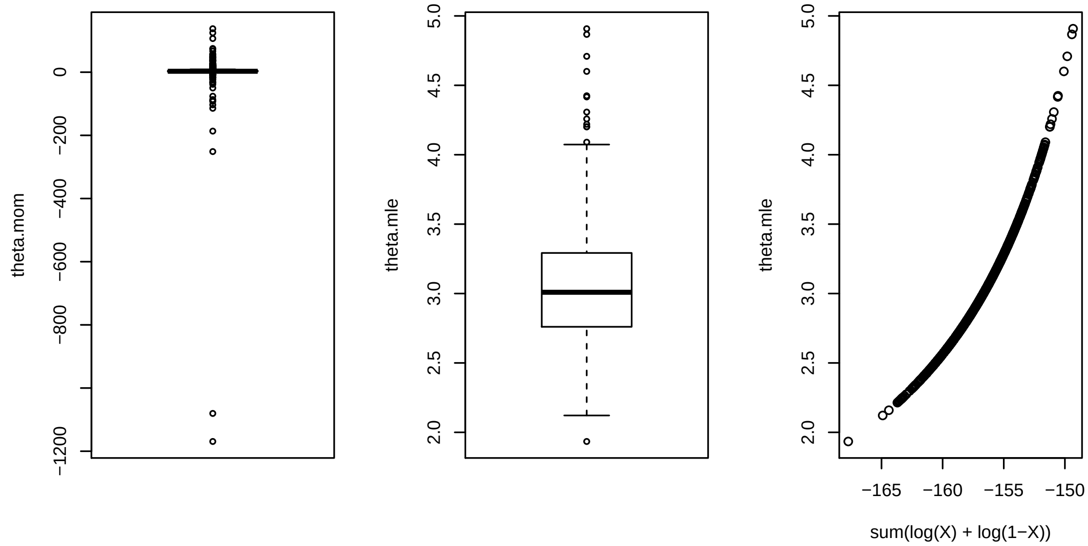
```
X <- c(0.39, 0.48, 0.78, 0.49, 0.52, 0.66, 0.44, 0.43, 0.08, 0.25)
n <- length(X)
theta.seq <- seq(1,6,length=201)
ll <- n*log(gamma(2*theta.seq))-2*n*log(gamma(theta.seq))+(theta.seq-1)*sum(log(X)+log(1-X))
plot(ll ~ theta.seq,type="l")
```

Give your best guess of the value of the MLE $\hat{\theta}$ of θ by looking at the plot.

(f) Compute the value of the MoMs estimator on the same data.

(g) What is wrong with the value of the MoMs estimator?

(h) A simulation was run in which $1{,}000$ random samples of size 100 were drawn from the $\text{Beta}(3, 3)$ distribution, and the MLE and MoMs estimator were computed on each data set. The figure below shows boxplots of the $1{,}000$ values of the MLE and the MoMs estimator as well as a plot of the values of the MLE versus the value of $\log\left(\prod_{i=1}^n X_i(1 - X_i)\right)$ over the $1{,}000$ simulations.



i. Which estimator, the MLE or the MoMs estimator, appears more reliable?
ii. What is the significance of the plot on the far right? Why would I show it?

4. Suppose you observe the following independent realizations of a random variable X:

```
0.46 0.64 0.59 0.93 1.63 0.61 1.75 0.61 0.78 0.81
1.00 0.59 0.77 1.36 1.79 1.15 0.44 0.81 0.71 0.34
```

(a) Assuming that X follows a Gamma distribution, find the MoMs and the MLEs of (α, β). Use R as in the *Lec 11 R code* to find the MLEs.

(b) Make a histogram of the realizations of X and overlay the pdfs of the Gamma distributions under the MoMs estimators and the MLEs of (α, β). Refer to the *Lec 11 R code*.

(c) Comment on the difference between the MoMs estimator and the MLE.

(d) Find the MLE for $P(X < 1)$.

5. Suppose you observe the following independent realizations of a random variable X:

```
8.33  8.20 11.35  3.05 12.62  9.20 15.68 12.52  4.98 10.44
9.21  8.49  5.14  3.55  5.78  3.97 15.75  7.92  5.02 6.75
```

Assume that X follows a distribution with the density

$$f_X(x; a, b) = \frac{a}{b}\left(\frac{x}{b}\right)^{a-1} \exp\left[-\left(\frac{x}{b}\right)^{a}\right] \mathbb{1}(x > 0).$$

(a) Write down the likelihood function for a and b based on $X_1, \ldots, X_n$.

(b) Write down the log-likelihood function for a and b based on $X_1, \ldots, X_n$.

(c) Find the MLEs of (a, b). Use R as in the *Lec 11 R code* to find the MLEs.

(d) Find the MLE of b^2.

Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.:

- 9.74, 9.75, 9.78
- 9.80, 9.81, 9.82, 9.92