

# STAT 512 sp 2018 Exam II

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*Do not open this test until told to do so; no calculators allowed; no notes allowed; no books allowed; show your work so that partial credit may be given.*

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & \text{for } y \in \mathcal{Y} \\ 0 & \text{for } y \notin \mathcal{Y} \end{cases}$$

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} [F_X(x)]^{k-1} [1 - F_X(x)]^{n-k} f_X(x)$$

$$X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Normal}(\mu, \sigma^2) \implies \begin{cases} (n-1)S_n^2/\sigma^2 \sim \chi_{n-1}^2 \\ \sqrt{n}(\bar{X}_n - \mu)/\sigma \sim \text{Normal}(0, 1) \\ \sqrt{n}(\bar{X}_n - \mu)/S_n \sim t_{n-1} \end{cases}$$

$$W_1 \sim \chi_{\nu_1}^2, \quad W_2 \sim \chi_{\nu_2}^2, \quad W_1, W_2 \text{ indep.} \implies (W_1/\nu_1)/(W_2/\nu_2) \sim F_{\nu_1, \nu_2}$$

pmf/pdf	$\mathcal{X}$	$M_X(t)$	$\mathbb{E}X$	$\text{Var } X$
$p_X(x; p) = p^x(1-p)^{1-x},$	$x = 0, 1$	$pe^t + (1-p)$	$p$	$p(1-p)$
$p_X(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x},$	$x = 0, 1, \dots, n$	$[pe^t + (1-p)]^n$	$np$	$np(1-p)$
$p_X(x; p) = (1-p)^{x-1}p,$	$x = 1, 2, \dots$	$\frac{pe^t}{1-(1-p)e^t}$	$p^{-1}$	$(1-p)p^{-2}$
$p_X(x; p, r) = \binom{x-1}{r-1} (1-p)^{x-r} p^r,$	$x = r, r+1, \dots$	$\left[ \frac{pe^t}{1-(1-p)e^t} \right]^r$	$rp^{-1}$	$r(1-p)p^{-2}$
$p_X(x; \lambda) = e^{-\lambda} \lambda^x / x!$	$x = 0, 1, \dots$	$e^{\lambda(e^t-1)}$	$\lambda$	$\lambda$
$p_X(x; N, M, K) = \binom{M}{x} \binom{N-M}{K-x} / \binom{N}{K}$	$x = 0, 1, \dots, K$	¡complicadísimo!		$\frac{KM}{N} \frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x; K) = \frac{1}{K}$	$x = 1, \dots, K$	$\frac{1}{K} \sum_{x=1}^K e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x; x_1, \dots, x_n) = \frac{1}{n}$	$x = x_1, \dots, x_n$	$\frac{1}{n} \sum_{i=1}^n e^{tx_i}$	$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$	$\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$
$f_X(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	$\mu$	$\sigma^2$
$f_X(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x; \lambda) = \frac{1}{\lambda} \exp\left(-\frac{x}{\lambda}\right)$	$0 < x < \infty$	$(1-\lambda t)^{-1}$	$\lambda$	$\lambda^2$
$f_X(x; \nu) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}} x^{\nu/2-1} \exp\left(-\frac{x}{2}\right)$	$0 < x < \infty$	$(1-2t)^{-\nu/2}$	$\nu$	$2\nu$
$f_X(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \left( \prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r} \right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

The table below gives some values of the function  $\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ :

$z$	1.282	1.645	1.96	2.576
$\Phi(z)$	0.9	0.95	0.975	0.995

1. Let  $X_1, \dots, X_5$  be independent rvs from the  $\text{Normal}(\mu, \sigma^2)$  distribution and let  $Z_1, \dots, Z_5$  be independent rvs from the  $\text{Normal}(0, 1)$  distribution. For each distribution (a)–(h), give the number of the quantity among the quantities (1)–(8) which has that distribution.

(a)  $\text{Normal}(0, 1)$

(b)  $F_{2,1}$

(c)  $\chi_1^2$

(d)  $F_{3,2}$

(e)  $t_1$

(f)  $\chi_3^2$

(g)  $\chi_2^2$

(h)  $t_2$

$$\left( \frac{X_1 - \mu}{\sigma} \right)^2 \quad (1)$$

$$\left( \frac{X_1 - \mu}{\sigma} \right)^2 + \left( \frac{X_2 - \mu}{\sigma} \right)^2 + \left( \frac{X_3 - \mu}{\sigma} \right)^2 \quad (2)$$

$$\frac{Z_1}{\sqrt{Z_2^2}} \quad (3)$$

$$\frac{1}{\sigma^2} \left( \left[ X_1 - \left( \frac{X_1 + X_2 + X_3}{3} \right) \right]^2 + \left[ X_2 - \left( \frac{X_1 + X_2 + X_3}{3} \right) \right]^2 + \left[ X_3 - \left( \frac{X_1 + X_2 + X_3}{3} \right) \right]^2 \right) \quad (4)$$

$$\frac{\left( [X_1 - (\frac{X_1 + X_2 + X_3}{3})]^2 + [X_2 - (\frac{X_1 + X_2 + X_3}{3})]^2 + [X_3 - (\frac{X_1 + X_2 + X_3}{3})]^2 \right)}{2 \left( [X_4 - (\frac{X_4 + X_5}{2})]^2 + [X_5 - (\frac{X_4 + X_5}{2})]^2 \right)} \quad (5)$$

$$\frac{(X_1 + X_2 + X_3 + X_4 + X_5)/5 - \mu}{\sigma/\sqrt{5}} \quad (6)$$

$$\frac{(X_1 + X_2 + X_3)/3 - \mu}{(\frac{1}{2} \sum_{i=1}^3 [X_i - (X_1 + X_2 + X_3)/3]^2)^{1/2} / \sqrt{3}} \quad (7)$$

$$\frac{(Z_1^2 + Z_2^2 + Z_3^2)/3}{(Z_4^2 + Z_5^2)/2} \quad (8)$$

2. Let  $X_1, \dots, X_n$  be independent rvs with the same distribution as the rv  $X$ , which has the cdf

$$F_X(x) = \begin{cases} 1 - e^{-x\gamma} & x > 0 \\ 0 & x \leq 0, \end{cases}$$

for some constant  $\gamma > 0$ .

- (a) Find the pdf of  $X$ .
- (b) Find the pdf of the first order statistic  $X_{(1)}$ .
- (c) The quantity  $Y = X_1 + \dots + X_n$  has a Gamma distribution; find the parameters  $\alpha$  and  $\beta$  for which  $Y \sim \text{Gamma}(\alpha, \beta)$ . *Hint: Get the mgf of  $Y$ .*
- (d) Find  $\mathbb{E}[1/Y]$ .
- (e) Let  $\hat{\gamma} = n/Y$  be an estimator of  $\gamma$  and compute  $\text{Bias } \hat{\gamma}$ .

3. Let  $X_1, \dots, X_7$  be a random sample from a distribution with pdf

$$f_X(x) = \frac{1}{b-a} \mathbb{1}(a < x < b), \quad \text{for } a, b \in \mathbb{R}, a < b.$$

- (a) Find the cdf  $F_X$  corresponding to the pdf  $f_X$ .
- (b) Find the pdf of  $X_{(4)}$ .
- (c) Find the pdf of the transformation  $Y = (X_{(4)} - a)/(b - a)$  of  $X_{(4)}$  and give the name of the distribution of  $Y$ .
- (d) Give  $\mathbb{E}Y$  and  $\text{Var } Y$ .
- (e) Find the MSE of  $X_{(4)}$  when  $X_{(4)}$  is used as an estimator of  $(a+b)/2$ . *Hint:  $X_{(4)} = (b-a)Y + a$ .*