STAT 512 sp 2018 Exam III

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Do not open this test until told to do so; no calculators allowed; no notes allowed; no books allowed; show your work so that partial credit may be given.

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| & \text{for } y \in \mathcal{Y} \\ 0 & \text{for } y \notin \mathcal{Y} \end{cases}$$

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}[F_X(x)]^{k-1}[1-F_X(x)]^{n-k}f_X(x)$$

$$X_1,\dots,X_n \overset{\text{iid}}{\sim} \text{Normal}(\mu,\sigma^2) \implies \begin{cases} (n-1)S_n^2/\sigma^2 \sim \chi^2_{n-1} \\ \sqrt{n}(\bar{X}_n-\mu)/\sigma \sim \text{Normal}(0,1) \\ \sqrt{n}(\bar{X}_n-\mu)/S_n \sim t_{n-1} \end{cases}$$

$$W_1 \sim \chi^2_{\nu_1}, \quad W_2 \sim \chi^2_{\nu_2}, \quad W_1, W_2 \text{ indep.} \implies (W_1/\nu_1)/(W_2/\nu_2) \sim F_{\nu_1,\nu_2}$$

pmf/pdf	$\mathcal{X}$	$M_X(t)$	$\mathbb{E}X$	$\text{Var}\,X$
$p_X(x;p) = p^x(1-p)^{1-x}$,	$x = 0,1$	$pe^t + (1-p)$	p	$p(1-p)$
$p_X(x;n,p) = \binom{n}{x}p^x(1-p)^{n-x}$,	$x = 0,1,\dots,n$	$[pe^t + (1-p)]^n$	np	$np(1-p)$
$p_X(x;p) = (1-p)^{x-1}p$,	$x = 1,2,\dots$	$\frac{pe^t}{1-(1-p)e^t}$	p^{-1}	$(1-p)p^{-2}$
$p_X(x;p,r) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$,	$x = r, r+1,\dots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	rp^{-1}	$r(1-p)p^{-2}$
$p_X(x;\lambda) = e^{-\lambda}\lambda^x/x!$	$x = 0,1,\dots$	$e^{\lambda(e^t-1)}$	λ	λ
$p_X(x;N,M,K) = \binom{M}{x}\binom{N-M}{K-x}/\binom{N}{K}$	$x = 0,1,\dots,K$	¡complicadísimo!	$\frac{KM}{N}$	$\frac{KM}{N}\frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x;K) = \frac{1}{K}$	$x = 1,\dots,K$	$\frac{1}{K}\sum_{x=1}^K e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x;x_1,\dots,x_n) = \frac{1}{n}$	$x = x_1,\dots,x_n$	$\frac{1}{n}\sum_{i=1}^n e^{tx_i}$	$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$	$\frac{1}{n}\sum_{i=1}^n(x_i-\bar{x})^2$
$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2
$f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x;\lambda) = \frac{1}{\lambda}\exp\left(-\frac{x}{\lambda}\right)$	$0 < x < \infty$	$(1-\lambda t)^{-1}$	λ	λ^2
$f_X(x;\nu) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}}x^{\nu/2-1}\exp\left(-\frac{x}{2}\right)$	$0 < x < \infty$	$(1-2t)^{-\nu/2}$	ν	2ν
$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^\infty \frac{t^k}{k!}\left(\prod_{r=0}^k \frac{\alpha+r}{\alpha+\beta+r}\right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

The table below gives some values of the function $\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$:

z	1.282	1.645	1.96	2.576
$\Phi(z)$	0.9	0.95	0.975	0.995

1. Let $X_1, \ldots, X_n$ be a random sample from the Uniform$(-\theta, \theta)$ distribution for some $\theta > 0$.

   (a) Find the cdf of the Uniform$(-\theta, \theta)$ distribution.

   (b) Find the pdf of the largest order statistic $X_{(n)}$.

   (c) Show that $X_{(n)}$ is a consistent estimator of θ.
   Recall that there are two ways to show consistency: You can show that the MSE goes to zero as $n \to \infty$ or you can show that $P(|X_{(n)} - \theta| < \epsilon)$ approaches a certain limit for every $\epsilon > 0$ as $n \to \infty$.

   (d) The statistic $X_{(n)}$ is not a sufficient statistic for θ in this example. Give a heuristic argument for why this is so *(A heuristic argument is an appeal to intuition without rigorous proof).*

2. Let Y_n be a sequence of random variables such that the cdf of Y_n is given by

$$F_{Y_n}(y) = \begin{cases} (1 - e^{-y}/n)^n, & y > -\log n \\ 0, & y \leq -\log n \end{cases}$$

   for all $n \geq 1$.

   (a) Write an expression for $P(-1 < Y_n < 1)$ using the cdf F_{Y_n}.

   (b) The sequence of random variables Y_n converges in distribution to a random variable Y. What is the cdf of Y? *Recall that* $\lim_{n\to\infty}(1 + a/n)^n = e^a$ *for any* $a \in \mathbb{R}$.

   (c) Write an expression for $\lim_{n\to\infty} P(-1 < Y_n < 1)$.

3. Let $X_1, \ldots, X_n$ be a random sample from a distribution with pdf

$$f_X(x; \lambda) = \lambda^{-1} \exp(-x\lambda^{-1}) \mathbb{1}(x > 0)$$

   for some $\lambda > 0$ and let $\bar{X}_n = n^{-1}(X_1 + \cdots + X_n)$ and $S_n^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$.

   (a) Use mgfs to identify the distribution of $\bar{X}_n$.

   (b) Write down the probability

$$P\left(\bar{X}_n - 1.96\frac{\lambda}{\sqrt{n}} < \lambda < \bar{X}_n + 1.96\frac{\lambda}{\sqrt{n}}\right)$$

   as an integral over the pdf of $\bar{X}_n$.
   Hint: First rearrange the probability statement so that $\bar{X}_n$ is "in the middle".

   (c) Give the limit as $n \to \infty$ of your integral in part (b).

   (d) Does $(n-1)S_n^2/\lambda^2$ have the χ^2_{n-1} distribution? Explain why it does or does not.

4. Let $X_1, \ldots, X_n$ be a random sample from the Gamma$(4, 2)$ distribution and let $\bar{X}_n = n^{-1}(X_1 + \cdots + X_n)$ and $S_n^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$.

   (a) Give a function of $\bar{X}_n$ which converges in distribution to a standard Normal random variable.

   (b) Give $\lim_{n\to\infty} P(15 < S_n^2 < 17)$.

   (c) Give $\lim_{n\to\infty} P(4 < \bar{X}_n < 5)$.