STAT 512 sp 2018 Final Exam

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Do not open this test until told to do so; no calculators allowed; no notes allowed; no books allowed; show your work so that partial credit may be given.

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| & \text{for } y \in \mathcal{Y} \\ 0 & \text{for } y \notin \mathcal{Y} \end{cases}$$

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}[F_X(x)]^{k-1}[1-F_X(x)]^{n-k}f_X(x)$$

$$X_1,\dots,X_n \overset{\text{iid}}{\sim} \text{Normal}(\mu,\sigma^2) \implies \begin{cases} (n-1)S_n^2/\sigma^2 \sim \chi^2_{n-1} \\ \sqrt{n}(\bar{X}_n-\mu)/\sigma \sim \text{Normal}(0,1) \\ \sqrt{n}(\bar{X}_n-\mu)/S_n \sim t_{n-1} \end{cases}$$

$$W_1 \sim \chi^2_{\nu_1}, \quad W_2 \sim \chi^2_{\nu_2}, \quad W_1, W_2 \text{ indep.} \implies (W_1/\nu_1)/(W_2/\nu_2) \sim F_{\nu_1,\nu_2}$$

pmf/pdf	$\mathcal{X}$	$M_X(t)$	$\mathbb{E}X$	$\text{Var}\,X$
$p_X(x;p) = p^x(1-p)^{1-x}$,	$x = 0,1$	$pe^t + (1-p)$	p	$p(1-p)$
$p_X(x;n,p) = \binom{n}{x}p^x(1-p)^{n-x}$,	$x = 0,1,\dots,n$	$[pe^t + (1-p)]^n$	np	$np(1-p)$
$p_X(x;p) = (1-p)^{x-1}p$,	$x = 1,2,\dots$	$\frac{pe^t}{1-(1-p)e^t}$	p^{-1}	$(1-p)p^{-2}$
$p_X(x;p,r) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$,	$x = r, r+1, \dots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	rp^{-1}	$r(1-p)p^{-2}$
$p_X(x;\lambda) = e^{-\lambda}\lambda^x/x!$	$x = 0,1,\dots$	$e^{\lambda(e^t-1)}$	λ	λ
$p_X(x;N,M,K) = \binom{M}{x}\binom{N-M}{K-x}/\binom{N}{K}$	$x = 0,1,\dots,K$	¡complicadísimo!	$\frac{KM}{N}$	$\frac{KM}{N}\frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x;K) = \frac{1}{K}$	$x = 1,\dots,K$	$\frac{1}{K}\sum_{x=1}^K e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x;x_1,\dots,x_n) = \frac{1}{n}$	$x = x_1,\dots,x_n$	$\frac{1}{n}\sum_{i=1}^n e^{tx_i}$	$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$	$\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$
$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2
$f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x;\lambda) = \frac{1}{\lambda}\exp\left(-\frac{x}{\lambda}\right)$	$0 < x < \infty$	$(1-\lambda t)^{-1}$	λ	λ^2
$f_X(x;\nu) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}}x^{\nu/2-1}\exp\left(-\frac{x}{2}\right)$	$0 < x < \infty$	$(1-2t)^{-\nu/2}$	ν	2ν
$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^\infty \frac{t^k}{k!}\left(\prod_{r=0}^k \frac{\alpha+r}{\alpha+\beta+r}\right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

The table below gives some values of the function $\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$:

z	1.282	1.645	1.96	2.576
$\Phi(z)$	0.9	0.95	0.975	0.995

1. Let $X_1, \ldots, X_3$ be independent rvs from the Normal(μ, σ^2) distribution and let $Z_1, \ldots, Z_4$ be independent rvs from the Normal$(0, 1)$ distribution. Give the distributions of the following quantities:

   (a)
$$\frac{Z_1}{\sqrt{(Z_2^2 + Z_3^2 + Z_4^2)/3}}$$

   (b)
$$\frac{(X_1 + X_2 + X_3)/3 - \mu}{\sigma/\sqrt{3}}$$

   (c)
$$\frac{(X_1 + X_2 + X_3)/3 - \mu}{(\frac{1}{2}\sum_{i=1}^{3}[X_i - (X_1 + X_2 + X_3)/3]^2)^{1/2}/\sqrt{3}}$$

   (d)
$$\frac{Z_1^2 + Z_2^2}{Z_3^2 + Z_4^2}$$

   (e)
$$\sum_{i=1}^{3}[X_i - (X_1 + X_2 + X_3)/3]^2/\sigma^2$$

2. Let $X_1, \ldots, X_7$ be independent random variables such that $X_i \sim$ Poisson(λ_i), $i = 1, \ldots, 7$, with
$$\lambda_1 = 3, \quad \lambda_2 = 3, \quad \lambda_3 = 4, \quad \lambda_4 = 5, \quad \lambda_5 = 6, \quad \lambda_6 = 6, \quad \lambda_7 = 3.$$

   (a) Give the distribution of the random variable $Y = X_1 + \cdots + X_7$.

   (b) Supposing that $X_1, \ldots, X_7$ represent the number of car accidents occurring on each day during one week in some city, write an expression for the probability that the total number of car accidents in the city during that week exceeds thirty.

3. As the result of conducting several independent Bernoulli trials, you observe
$$1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0$$

   (a) What is the value of the success probability p under which the observed sequence of successes and failures has the highest probability of occurrence?

   (b) What is the value of the success probability p under which the Bernoulli(p) distribution has the same mean as your sample of ones and zeros?

   (c) Suppose the true success probability is 1/3. Give a function of
$$\hat{p} = \frac{\#\{\text{successes}\}}{\#\{\text{trials}\}}$$
   which converges in distribution to a standard Normal random variable as the number of trials is increased to infinity.

(d) Suppose you conduct 100 trials and observe 32 successes. For any $\alpha \in (0, 1)$, give an expression for an approximate $(1 - \alpha)100\%$ confidence interval for the unknown success probability p.

4. Let $X \sim f_X(x) = (1/2)\mathbb{1}(-1 < x < 1)$ and let $Y = |X|$.

   (a) Find $P_Y(Y \leq 1/2)$.

   (b) Give the cdf of Y.

   (c) Give the pdf of Y.

5. Let $X_1, \ldots, X_n$ be a random sample from the distribution with pdf

$$f_X(x) = \begin{cases} \alpha 5^{\alpha} x^{-(\alpha+1)} & x \geq 5 \\ 0 & x < 5 \end{cases}$$

   where $\alpha > 0$ is an unknown parameter.

   (a) Write down the log-likelihood function for α.

   (b) Find the MLE of α.

6. Let $X_1, \ldots, X_n$ be a random sample from the distribution with pdf given by

$$f_X(x; \theta) = (1 - \theta)^{-1}\mathbb{1}(\theta \leq x \leq 1)$$

   for some $\theta < 1$. Let $\bar{X}_n = n^{-1}(X_1 + \cdots + X_n)$.

   (a) Let $\bar{\theta}$ denote the MoMs estimator of θ. Find $\bar{\theta}$.

   (b) Find the bias of $\bar{\theta}$.

   (c) Find the MSE of $\bar{\theta}$.

   (d) Argue whether or not $\bar{\theta}$ is a consistent estimator for θ.

   (e) Argue whether or not there exists a better estimator of θ than $\bar{\theta}$.

   (f) Give the cdf corresponding to the pdf $f_X(x; \theta)$.

   (g) Let $\hat{\theta}$ denote the MLE of θ. Find $\hat{\theta}$. *Hint: You cannot use calculus methods to find it.*

   (h) Find the pdf of $\hat{\theta}$.

   (i) The estimator

$$\hat{\theta}^{\text{unbiased}} = \left(\hat{\theta} - \frac{1}{n+1}\right)\left(\frac{n+1}{n}\right),$$

   where $\hat{\theta}$ is the MLE of θ, is an unbiased estimator of θ. Argue that this estimator is the MVUE.