

STAT 512 sp 2020 Exam I

Karl B. Gregory

Do not open this test until told to do so; no calculators allowed; no notes allowed; no books allowed; show your work so that partial credit may be given.

pmf/pdf	\mathcal{X}	$M_X(t)$	EX	$\text{Var } X$
$p_X(x; p) = p^x(1-p)^{1-x},$	$x = 0, 1$	$pe^t + (1-p)$	p	$p(1-p)$
$p_X(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x},$	$x = 0, 1, \dots, n$	$[pe^t + (1-p)]^n$	np	$np(1-p)$
$p_X(x; p) = (1-p)^{x-1}p,$	$x = 1, 2, \dots$	$\frac{pe^t}{1-(1-p)e^t}$	p^{-1}	$(1-p)p^{-2}$
$p_X(x; p, r) = \binom{x-1}{r-1}(1-p)^{x-r}p^r,$	$x = r, r+1, \dots$	$\left[\frac{pe^t}{1-(1-p)e^t} \right]^r$	rp^{-1}	$r(1-p)p^{-2}$
$p_X(x; \lambda) = e^{-\lambda}\lambda^x/x!$	$x = 0, 1, \dots$	$e^{\lambda(e^t-1)}$	λ	λ
$p_X(x; N, M, K) = \binom{M}{x} \binom{N-M}{K-x} / \binom{N}{K}$	$x = 0, 1, \dots, K$	¡complicadísimo!		$\frac{KM}{N} \frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x; K) = \frac{1}{K}$	$x = 1, \dots, K$	$\frac{1}{K} \sum_{x=1}^K e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x; x_1, \dots, x_n) = \frac{1}{n}$	$x = x_1, \dots, x_n$	$\frac{1}{n} \sum_{i=1}^n e^{tx_i}$	$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$	$\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$
$f_X(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2
$f_X(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x; \lambda) = \frac{1}{\lambda} \exp\left(-\frac{x}{\lambda}\right)$	$0 < x < \infty$	$(1-\lambda t)^{-1}$	λ	λ^2
$f_X(x; \nu) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}} x^{\nu/2-1} \exp\left(-\frac{x}{2}\right)$	$0 < x < \infty$	$(1-2t)^{-\nu/2}$	ν	2ν
$f_X(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \left(\prod_{r=0}^k \frac{\alpha+r}{\alpha+\beta+r} \right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \text{ for } y \in \mathcal{Y}$$

$$f_{(Y_1, Y_2)}(y_1, y_2) = f_{(X_1, X_2)}(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)) |J(y_1, y_2)| \quad \text{for } (y_1, y_2) \in \mathcal{Y}$$

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} [F_X(x)]^{k-1} [1 - F_X(x)]^{n-k} f_X(x)$$

$$X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2) \implies \begin{cases} (n-1)S_n^2/\sigma^2 \sim \chi_{n-1}^2 \\ \sqrt{n}(\bar{X}_n - \mu)/\sigma \sim \text{Normal}(0, 1) \\ \sqrt{n}(\bar{X}_n - \mu)/S_n \sim t_{n-1} \end{cases}$$

$$W_1 \sim \chi_{\nu_1}^2, \quad W_2 \sim \chi_{\nu_2}^2, \quad W_1, W_2 \text{ indep.} \implies (W_1/\nu_1)/(W_2/\nu_2) \sim F_{\nu_1, \nu_2}$$

1. Let X be a rv with cdf given by $F_X(x) = x^2/\pi^2$ for $x \in (0, \pi)$ and let $Y = -\log(X/\pi)$.
 - (a) Give the pdf of X .
 - (b) Give a transformation of X which will have the Uniform(0, 1) distribution.
 - (c) Give the transformation of a Uniform(0, 1) random variable which can be used to generate a realization of the random variable X .
 - (d) Let X_1, \dots, X_5 be independent rvs with the same distribution as X . Give the pdf of $X_{(3)}$.
 - (e) Give the support of Y .
 - (f) Find the pdf of Y .
2. Let X and Y be independent rvs such that $X \sim \text{Poisson}(3)$ and $Y \sim \text{Poisson}(5)$ and let $U = X + Y$.
 - (a) Give the support of U .
 - (b) Give the mgf of U .
 - (c) Write down the pmf of U .
3. The order statistics $U = X_{(k)}$ and $V = X_{(k+1)}$ of a random sample $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Uniform}(0, 1)$ have joint pdf given by

$$f_{U,V}(u, v) = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k)} u^{k-1} (1-v)^{(n-k)-1} \quad \text{for } 0 < u < v < 1$$

for each $k = 1, \dots, n-1$. Let $R = U/V$ and $M = V$.

 - (a) State whether U and V are independent and explain how you determined your answer.
 - (b) Give the support of the random variable pair (R, M) .
 - (c) Give the Jacobian of the transformation.
 - (d) Find the joint pdf of R and M .
 - (e) State whether R and M are independent and explain how you determined your answer.
 - (f) Describe in words how you would obtain the marginal pdf of R .
4. Let W_1, W_2, W_3 and Z_1, Z_2, Z_3, Z_4 be independent rvs such that $Z_i \sim \text{Normal}(0, 1)$ for $i = 1, \dots, 4$ and $W_i \sim \chi^2_1$ for $i = 1, 2, 3$. Let $\bar{Z} = (1/4) \sum_{i=1}^4 Z_i$. Determine the distributions of the following:
 - (a) $2\bar{Z}$
 - (b) $W_1 + W_2 + W_3$
 - (c) $(1/3)(Z_1^2 + Z_2^2 + Z_3^2)/W_1$
 - (d) $\sum_{i=1}^4 (Z_i - \bar{Z})^2$
 - (e) $\sqrt{4\bar{Z}}/\sqrt{(W_1 + W_2)/2}$