STAT 512 sp 2020 Exam I

Karl B. Gregory

Do not open this test until told to do so; no calculators allowed; no notes allowed; no books allowed; show your work so that partial credit may be given.

pmf/pdf	$\mathcal{X}$	$M_X(t)$	$\mathbb{E}X$	Var X
$p_X(x;p) = p^x(1-p)^{1-x}$,	$x = 0, 1$	$pe^t + (1-p)$	p	$p(1-p)$
$p_X(x;n,p) = \binom{n}{x}p^x(1-p)^{n-x}$,	$x = 0, 1, \dots, n$	$[pe^t + (1-p)]^n$	np	$np(1-p)$
$p_X(x;p) = (1-p)^{x-1}p$,	$x = 1, 2, \dots$	$\frac{pe^t}{1-(1-p)e^t}$	p^{-1}	$(1-p)p^{-2}$
$p_X(x;p,r) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$,	$x = r, r+1, \dots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	rp^{-1}	$r(1-p)p^{-2}$
$p_X(x;\lambda) = e^{-\lambda}\lambda^x/x!$	$x = 0, 1, \dots$	$e^{\lambda(e^t-1)}$	λ	λ
$p_X(x;N,M,K) = \binom{M}{x}\binom{N-M}{K-x}/\binom{N}{K}$	$x = 0, 1, \dots, K$	¡complicadísimo!	$\frac{KM}{N}$	$\frac{KM}{N}\frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x;K) = \frac{1}{K}$	$x = 1, \dots, K$	$\frac{1}{K}\sum_{x=1}^K e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x;x_1,\dots,x_n) = \frac{1}{n}$	$x = x_1, \dots, x_n$	$\frac{1}{n}\sum_{i=1}^n e^{tx_i}$	$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$	$\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$
$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2
$f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x;\lambda) = \frac{1}{\lambda}\exp\left(-\frac{x}{\lambda}\right)$	$0 < x < \infty$	$(1-\lambda t)^{-1}$	λ	λ^2
$f_X(x;\nu) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}}x^{\nu/2-1}\exp\left(-\frac{x}{2}\right)$	$0 < x < \infty$	$(1-2t)^{-\nu/2}$	ν	2ν
$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^\infty \frac{t^k}{k!}\left(\prod_{r=0}^k \frac{\alpha+r}{\alpha+\beta+r}\right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| \text{ for } y \in \mathcal{Y}$$

$$f_{(Y_1,Y_2)}(y_1,y_2) = f_{(X_1,X_2)}(g_1^{-1}(y_1,y_2), g_2^{-1}(y_1,y_2))|J(y_1,y_2)| \quad \text{for } (y_2,y_2) \in \mathcal{Y}$$

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}[F_X(x)]^{k-1}[1-F_X(x)]^{n-k}f_X(x)$$

$$X_1,\dots,X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu,\sigma^2) \implies \begin{cases} (n-1)S_n^2/\sigma^2 \sim \chi^2_{n-1} \\ \sqrt{n}(\bar{X}_n - \mu)/\sigma \sim \text{Normal}(0,1) \\ \sqrt{n}(\bar{X}_n - \mu)/S_n \sim t_{n-1} \end{cases}$$

$$W_1 \sim \chi^2_{\nu_1}, \quad W_2 \sim \chi^2_{\nu_2}, \quad W_1, W_2 \text{ indep.} \implies (W_1/\nu_1)/(W_2/\nu_2) \sim F_{\nu_1,\nu_2}$$

1. Let X be a rv with cdf given by $F_X(x) = x^2/\pi^2$ for $x \in (0, \pi)$ and let $Y = -\log(X/\pi)$.

(a) Give the pdf of X.

Solution: Taking the derivative of the cdf, we see that the pdf of X is given by

$$f_X(x) = 2x/\pi^2 \quad \text{for } 0 < x < \pi.$$

(b) Give a transformation of X which will have the Uniform$(0, 1)$ distribution.

Solution: Passing X through its own cdf will result in a Uniform$(0, 1)$ random variable. That is,

$$U = X^2/\pi^2 \sim \text{Uniform}(0, 1).$$

(c) Give the transformation of a Uniform$(0, 1)$ random variable which can be used to generate a realization of the random variable X.

Solution: Setting $U = X^2/\pi^2$ and solving for X gives

$$X = \pi\sqrt{U}.$$

So we can generate a realization of X from $U \sim \text{Uniform}(0, 1)$ with the transformation $X = \pi\sqrt{U}$.

(d) Let $X_1, \dots, X_5$ be independent rvs with the same distribution as X. Give the pdf of $X_{(3)}$.

Solution: We have

$$f_{X_{(3)}}(x) = \frac{5!}{(3-1)!(5-3)!}\left[\frac{x^2}{\pi^2}\right]^{3-1}\left[1 - \frac{x^2}{\pi^2}\right]^{5-3}\frac{2x}{\pi^2} = \frac{60x}{\pi^2}\left[\frac{x^2}{\pi^2}\right]^2\left[1 - \frac{x^2}{\pi^2}\right]^2$$

for $0 < x < \pi$.

(e) Give the support of Y.

Solution: We have $\mathcal{Y} = (0, \infty)$.

(f) Find the pdf of Y.

Solution: We have

$$y = -\log(x/\pi) =: g(x) \iff x = \pi e^{-y} =: g^{-1}(y), \quad \text{and} \quad \frac{d}{dy}g^{-1}(y) = -\pi e^{-y}.$$

By the transformation method we have

$$f_Y(y) = \frac{2}{\pi^2}\pi e^{-y}| - \pi e^{-y}| = 2e^{-2y} \quad \text{for } y > 0.$$

2. Let X and Y be independent rvs such that $X \sim \text{Poisson}(3)$ and $Y \sim \text{Poisson}(5)$ and let $U = X + Y$.
   (a) Give the support of U.

   Solution: The support of U is $\{0, 1, 2 \ldots\}$.

   (b) Give the mgf of U.

   Solution: The mgf of U is given by

$$M_U(t) = M_X(t)M_Y(t) = e^{3(e^t-1)}e^{5(e^t-1)} = e^{8(e^t-1)}.$$

   (c) Write down the pmf of U.

   Solution: The pmf of U is given by

$$p_U(u) = \frac{e^{-8}8^u}{u!} \quad \text{for } u = 0, 1, 2, \ldots$$

3. The order statistics $U = X_{(k)}$ and $V = X_{(k+1)}$ of a random sample $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(0, 1)$ have joint pdf given by

$$f_{U,V}(u, v) = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k)}u^{k-1}(1-v)^{(n-k)-1} \quad \text{for } 0 < u < v < 1$$

for each $k = 1, \ldots, n - 1$. Let $R = U/V$ and $M = V$.
   (a) State whether U and V are independent and explain how you determined your answer.

   Solution: They are *not* independent, since the support of one variable depends on the value of the other. As a result, there is no way to write the joint pdf as the product of the marginal pdfs. Although the expression looks like it may be factorable into the product of a function of only u and a function of only v, it is not possible if the joint pdf is written for all $(u, v) \in \mathbb{R} \times \mathbb{R}$. Writing

$$f_{U,V}(u, v) = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k)}u^{k-1}(1-v)^{(n-k)-1}\mathbf{1}(0 < u < v < 1) \quad \text{for } (u, v) \in \mathbb{R} \times \mathbb{R},$$

   where $\mathbf{1}(\cdot)$ is the indicator function, we see that we cannot find the factorization required to show independence of U and V.

(b) Give the support of the random variable pair (R, M).

Solution: We have $(R, M) \in (0, 1) \times (0, 1)$.

(c) Give the Jacobian of the transformation.

Solution: We have

$$\begin{array}{l} r = u/v =: g_1(u, v) \\ m = v =: g_2(u, v) \end{array} \iff \begin{array}{l} u = rm =: g_1^{-1}(r, m) \\ v = m =: g_2^{-1}(r, m) \end{array}$$

with Jacobian

$$J(x, y) = \begin{vmatrix} \frac{d}{dr}rm & \frac{d}{dm}rm \\ \frac{d}{dr}m & \frac{d}{dm}m \end{vmatrix} = \begin{vmatrix} m & r \\ 0 & 1 \end{vmatrix} = m.$$

(d) Find the joint pdf of R and M.

Solution: The joint pdf of R and M is given by

$$\begin{aligned} f_{R,M}(r, m) &= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k)}(rm)^{k-1}(1-m)^{(n-k)-1} \cdot |m| \\ &= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k)} r^{k-1} m^{(k+1)-1}(1-m)^{(n-k)-1} \quad \text{for } 0 < m < 1, 0 < r < 1. \end{aligned}$$

(e) State whether R and M are independent and explain how you determined your answer.

Solution: They *are* independent, since we can factor the joint pdf into the product of a function of only r and a function of only m.

(f) Describe in words how you would obtain the marginal pdf of R.

Solution: We could find the marginal pdf of R by taking the integral $\int_0^1 f_{R,M}(r, m)dm$. The result would be the marginal pdf of R. Another way we could find the marginal pdf of R is to play with factorizations of the joint pdf of R and M until we have found a way to write it as the product of two marginal pdfs that we recognize. We find that we may write

$$f_{R,M}(r, m) = \frac{\Gamma(k+1)}{\Gamma(k)\Gamma(1)} r^{k-1}(1-r)^{1-1} \frac{\Gamma(k+1+n-k)}{\Gamma(k+1)\Gamma(n-k)} m^{(k+1)-1}(1-m)^{(n-k)-1}$$

for $0 < m < 1$ and $0 < r < 1$, by which we see that $R \sim \text{Beta}(k, 1)$ and $M \sim \text{Beta}(k+1, n-k)$.

4. Let W_1, W_2, W_3 and Z_1, Z_2, Z_3, Z_4 be independent rvs such that $Z_i \sim \text{Normal}(0, 1)$ for $i = 1, \ldots, 4$ and $W_i \sim \chi_1^2$ for $i = 1, 2, 3$. Let $\bar{Z} = (1/4)\sum_{i=1}^4 Z_i$. Determine the distributions of the following:

(a) $2\bar{Z}$

Solution: This has the Normal$(0, 1)$ distribution.

(b) $W_1 + W_2 + W_3$

Solution: This has the χ^2_3 distribution.

(c) $(1/3)(Z_1^2 + Z_2^2 + Z_3^2)/W_1$

Solution: This has the $F_{3,1}$ distribution.

(d) $\sum_{i=1}^4 (Z_i - \bar{Z})^2$

Solution: This has the χ^2_3 distribution.

(e) $\sqrt{4}\bar{Z}/\sqrt{(W_1 + W_2)/2}$

Solution: This has the t_2 distribution.