STAT 512 sp 2020 Exam II

Karl B. Gregory

This is a take-home test due to the COVID-19 suspension of face-to-face instruction. Do not communicate with classmates about the exam until after its due date/time. You may

- *Use your notes and the lecture notes.*
- *Use books.*
- *NOT work together with others.*

Write all answers on blank sheets of paper; then take pictures and merge to a PDF. Upload a single PDF to Blackboard.

pmf/pdf	$\mathcal{X}$	$M_X(t)$	$\mathbb{E}X$	$\text{Var}\,X$
$p_X(x;p) = p^x(1-p)^{1-x}$,	$x = 0, 1$	$pe^t + (1-p)$	p	$p(1-p)$
$p_X(x;n,p) = \binom{n}{x}p^x(1-p)^{n-x}$,	$x = 0, 1, \dots, n$	$[pe^t + (1-p)]^n$	np	$np(1-p)$
$p_X(x;p) = (1-p)^{x-1}p$,	$x = 1, 2, \dots$	$\frac{pe^t}{1-(1-p)e^t}$	p^{-1}	$(1-p)p^{-2}$
$p_X(x;p,r) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$,	$x = r, r+1, \dots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	rp^{-1}	$r(1-p)p^{-2}$
$p_X(x;\lambda) = e^{-\lambda}\lambda^x/x!$	$x = 0, 1, \dots$	$e^{\lambda(e^t-1)}$	λ	λ
$p_X(x;N,M,K) = \binom{M}{x}\binom{N-M}{K-x}/\binom{N}{K}$	$x = 0, 1, \dots, K$	¡complicadísimo!	$\frac{KM}{N}$	$\frac{KM}{N}\frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x;K) = \frac{1}{K}$	$x = 1, \dots, K$	$\frac{1}{K}\sum_{x=1}^K e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x;x_1,\dots,x_n) = \frac{1}{n}$	$x = x_1, \dots, x_n$	$\frac{1}{n}\sum_{i=1}^n e^{tx_i}$	$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$	$\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$
$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2
$f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x;\lambda) = \frac{1}{\lambda}\exp\left(-\frac{x}{\lambda}\right)$	$0 < x < \infty$	$(1-\lambda t)^{-1}$	λ	λ^2
$f_X(x;\nu) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}}x^{\nu/2-1}\exp\left(-\frac{x}{2}\right)$	$0 < x < \infty$	$(1-2t)^{-\nu/2}$	ν	2ν
$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^\infty \frac{t^k}{k!}\left(\prod_{r=0}^k \frac{\alpha+r}{\alpha+\beta+r}\right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}[F_X(x)]^{k-1}[1-F_X(x)]^{n-k}f_X(x)$$

1. Copy down this sentence on your answer sheet and put your signature underneath: *I have not collaborated with any other student on this exam. The work I have presented is my own.*

2. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Gamma}(2, \beta)$.
   (a) Write down the pdf of the $\text{Gamma}(2, \beta)$ distribution.
   (b) Give $\mathbb{E}S_n^2$.
   (c) Give $\mathbb{E}\bar{X}_n$.
   (d) Give $\text{Var}\,\bar{X}_n$.
   (e) Propose an unbiased estimator $\hat{\beta}$ of β which is based on $\bar{X}_n$.
   (f) Find the variance of your estimator $\hat{\beta}$.
   (g) Find $\text{MSE}\,\hat{\beta}$.
   (h) Argue whether your estimator $\hat{\beta}$ is consistent for β.

3. Let $\hat{\theta}_n$ be an estimator of a parameter θ.
   (a) Explain in your own words what it means if $\hat{\theta}_n$ is a consistent estimator for θ.
   (b) Explain how you would go about checking whether $\hat{\theta}_n$ is a consistent estimator for θ.
   (c) Explain in your own words what it means if $\hat{\theta}_n$ is a biased estimator of θ.
   (d) Suppose the variance of $\hat{\theta}_n$ does not get smaller as n is increased. What does this mean in terms of whether the estimator is consistent?

4. Let $X_1, \dots, X_n$ be a random sample from some distribution with mean μ and variance $\sigma^2 < \infty$.
   (a) Find $\lim_{n\to\infty} P(|\bar{X}_n - \mu| < 2\sigma/\sqrt{n})$. You may use R to compute any probabilities (just write by hand any R code you used).
   (b) What theorem did you invoke in order to get your answer to the previous question?
   (c) Find $\lim_{n\to\infty} P(|\bar{X}_n - \mu| < 0.001 \cdot \sigma)$.
   (d) Is the probability $P(\mu - 1 < \bar{X}_n < \mu + 1)$ an increasing or a decreasing function of n? Explain your answer.
   (e) Give the form of a large-sample 98% confidence interval for μ.

5. Let X_1, X_2, X_3 be independent rvs with the $\text{Uniform}(\theta - 1, \theta + 1)$ distribution, with density given by

$$f_X(x) = \frac{1}{2}\mathbf{1}(\theta - 1 < x < \theta + 1).$$

   Consider the three estimators of θ given by

$$\begin{aligned} \hat{\theta} &= (X_1 + X_2 + X_3)/3 \\ \tilde{\theta} &= X_{(2)} \\ \check{\theta} &= (X_{(1)} + X_{(3)})/2, \end{aligned}$$

   where $X_{(1)} < X_{(2)} < X_{(3)}$ are the ordered values of X_1, X_2, X_3.

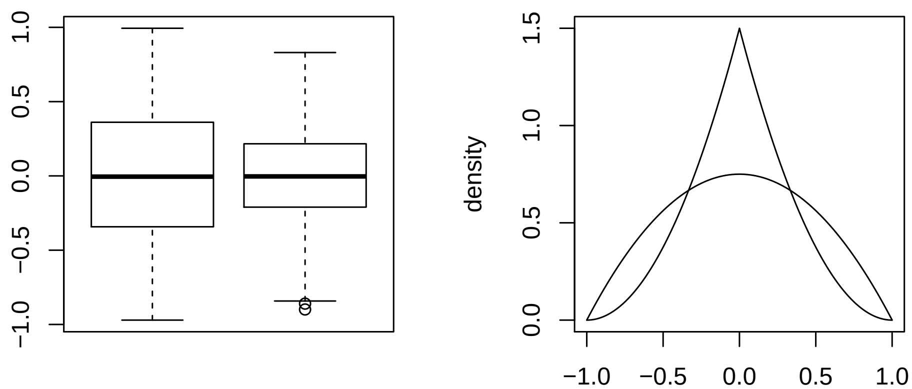
(a) Give $\mathbb{E}\hat{\theta}$.

(b) Using the fact that the variance of the Uniform(a, b) distribution is $(b-a)^2/12$, give $\operatorname{Var}\hat{\theta}$.

(c) Give $\operatorname{MSE}\hat{\theta}$.

(d) Find the pdf of $X_{(2)}$.

(e) It turns out that $\operatorname{Var}\tilde{\theta} = 1/5$ and $\operatorname{Var}\check{\theta} = 1/10$. A simulation was run in which 1,000 random samples of size 3 were drawn from the Uniform$(\theta - 1, \theta + 1)$ distribution under $\theta = 0$. The left panel of the figure shows boxplots of the 1,000 values of $\tilde{\theta}$ and $\check{\theta}$ and the right panel shows plots of the pdfs of these estimators under $\theta = 0$.



i. Comment on whether you think the estimators $\tilde{\theta}$ and $\check{\theta}$ are biased or unbiased.

ii. Of the three estimators $\hat{\theta}$, $\tilde{\theta}$, and $\check{\theta}$, choose the one you think is the best. Explain why you think it is the best.