STAT 512 sp 2020 Final Exam

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This is a take-home test due to the COVID-19 suspension of face-to-face instruction. Do not communicate with classmates about the exam until after its due date/time. You may

- *Use your notes and the lecture notes.*
- *Use books.*
- *NOT work together with others.*

Write all answers on blank sheets of paper; then take pictures and merge to a PDF. Upload a single PDF to Blackboard.

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}[F_X(x)]^{k-1}[1-F_X(x)]^{n-k}f_X(x)$$

$$X_1,\dots,X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu,\sigma^2) \implies \begin{cases} (n-1)S_n^2/\sigma^2 \sim \chi^2_{n-1} \\ \sqrt{n}(\bar{X}_n-\mu)/\sigma \sim \text{Normal}(0,1) \\ \sqrt{n}(\bar{X}_n-\mu)/S_n \sim t_{n-1} \end{cases}$$

$$W_1 \sim \chi^2_{\nu_1}, \quad W_2 \sim \chi^2_{\nu_2}, \quad W_1, W_2 \text{ indep.} \implies (W_1/\nu_1)/(W_2/\nu_2) \sim F_{\nu_1,\nu_2}$$

The table below gives some values of the function $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$:

z	0.841	1.282	1.645	1.96	2.326	2.576
$\Phi(z)$	0.80	0.90	0.95	0.975	.990	0.995

1. Copy down this sentence on your answer sheet and put your signature underneath: *I have not collaborated with any other student on this exam. The work I have presented is my own.*

2. Let $X_1, \ldots, X_n$ be a random sample from a population with pdf given by

$$f_X(x;\tau) = \sqrt{\frac{\tau}{\pi}} \exp(-\tau x^2), \quad -\infty < x < \infty,$$

for some $\tau > 0$, and note that the first two population moments are $\mu_1' = 0$ and $\mu_2' = 1/(2\tau)$.

   (a) Set $m_2' = \mu_2'$, where m_2' is the second sample moment, to find the MoMs estimator for τ.

   (b) Write down the likelihood function for τ based on $X_1, \ldots, X_n$.

   (c) Identify a sufficient statistic for τ.

   (d) Give the log-likelihood.

   (e) Find the MLE $\hat{\tau}$ for τ.

   (f) Let $Y = \sqrt{2\tau}X$, where $X \sim f_X(x;\tau)$. Use the transformation method to find the pdf of Y.

   (g) Identify the distribution of Y and give the distribution of $W = Y^2$.

   (h) Give the distribution of the random variable $S = \sum_{i=1}^n (\sqrt{2\tau}X_i)^2$.

   (i) Show that $\hat{\tau} = n\tau/S$.

   (j) We find that $\mathbb{E}[1/S] = 1/(n-2)$. Use this to find $\mathbb{E}\hat{\tau}$.

   (k) Propose an estimator $\hat{\tau}_{\text{unbiased}}$ based on $\hat{\tau}$ that is unbiased for τ.

   (l) Comment on whether there could exist an unbiased estimator besides $\hat{\tau}_{\text{unbiased}}$ with smaller variance.

3. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$.

   (a) Give the MoMs estimator $\bar{p}$ for p.

   (b) Give the variance of $\bar{p}$.

   (c) Give the MSE of $\bar{p}$.

   (d) Is the MoMs estimator a consistent estimator for p? Explain your answer.

   (e) Write down the joint pmf of $X_1, \ldots, X_n$.

   (f) Find a sufficient statistic for p.

   (g) Is the MoMs estimator of p the MVUE? Explain your answer.

4. Let $Z_1, Z_2, Z_3, Z_4, Z_5 \overset{\text{ind}}{\sim} \text{Normal}(0,1)$. Identify the distributions of the following.

   (a) $Z_1^2 + Z_2^2$

   (b) $\dfrac{(Z_1^2 + Z_2^2 + Z_3^2)/3}{(Z_4^2 + Z_5^2)/2}$

   (c) $\dfrac{1}{\sqrt{3}}(Z_1 + Z_2 + Z_3)$

(d) $\frac{1}{2}(Z_1 + Z_2) / \sqrt{\frac{1}{2}\left[\left(Z_1 - \left(\frac{Z_1 + Z_2}{2}\right)\right)^2 + \left(Z_2 - \left(\frac{Z_1 + Z_2}{2}\right)\right)^2\right]}$

5. Let $X_1, \ldots, X_n$ be a random sample from the distribution with cdf given by

$$F_X(x; B, M, \nu) = \frac{1}{[1 + \exp(-B(x - M))]^{1/\nu}}, \quad -\infty < x < \infty,$$

for some parameters $B > 0$, $-\infty < M < \infty$, $\nu > 0$.

(a) Find the function F_X^{-1} such that the random variable $F_X^{-1}(U)$ has the cdf F_X if $U \sim \text{Uniform}(0, 1)$.

(b) Give the pdf f_X corresponding to the cdf F_X.

(c) Write down the joint pdf of $X_1, \ldots, X_n$.

(d) Check whether $\sum_{i=1}^n X_i$ is a sufficient statistic for the parameter M. Explain your answer.

(e) Give an interval based on $X_1, \ldots, X_n$ which will contain the first population moment with probability approaching 0.99 as $n \to \infty$.