STAT 512 su 2021 Exam I

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This is a take-home test. Do not communicate with classmates about the exam until after its due date/time. You may

- *Use your notes and the lecture notes.*
- *Use books.*
- *NOT work together with others.*

Write all answers on blank sheets of paper; then take pictures and merge to a PDF. Upload a single PDF to Blackboard.

1. Copy down this sentence on your answer sheet and put your signature underneath: *I have not collaborated with any other student on this exam. The work I have presented is my own.*

2. Let X be the sum of two rolls of a die and let Y be the remainder when 12 is divided by X.

   (a) Give the support $\mathcal{X}$ of X.

   Solution: We have $\mathcal{X} = \{2, \dots, 12\}$.

   (b) Give the support $\mathcal{Y}$ of Y.

   Solution: We have $\mathcal{Y} = \{0, \dots, 5\}$.

   (c) Letting g represent the transformation from X to Y, give

   i. $g^{-1}(0)$

   Solution: We have $g^{-1}(0) = \{2, 3, 4, 6, 12\}$.

   ii. $g^{-1}(1)$.

   Solution: We have $g^{-1}(1) = \{11\}$.

   (d) Make a table giving the probability distribution of Y of the form

y	
$P(Y = y)$	

Solution:

y	0	1	2	3	4	5
$P(Y = y)$	12/36	2/36	7/36	4/36	5/36	6/36

We get this from

y	0	1	2	3	4	5
$g^{-1}(y)$	{2,3,4,6,12}	{11}	{5,10}	{9}	{8}	{7}
$P(Y = y) = P(X \in g^{-1}(y))$	12/36	2/36	7/36	4/36	5/36	6/36

We get the probabilities $P(X \in g^{-1}(y))$ from the probability distribution of X, which we can write as

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

3. Let $X \sim \text{Beta}(\alpha, \beta)$ and let $Y = \tan(\pi(X - 1/2))$.

   (a) Give the support $\mathcal{X}$ of X.

   Solution: We have $\mathcal{X} = (0, 1)$.

(b) Give the support $\mathcal{Y}$ of Y.

Solution: We have $\mathcal{Y} = (-\infty, \infty)$.

(c) Give the pdf of Y.

Solution: We have

$$g^{-1}(y) = \frac{\tan^{-1}(y)}{\pi} + \frac{1}{2} \quad \text{and} \quad \frac{d}{dy} g^{-1}(y) = \frac{1}{\pi} \frac{1}{1+y^2},$$

so that

$$f_Y(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left[\frac{\tan^{-1}(y)}{\pi} + \frac{1}{2}\right]^{\alpha-1} \left[1 - \left(\frac{\tan^{-1}(y)}{\pi} + \frac{1}{2}\right)\right]^{\beta-1} \frac{1}{\pi} \frac{1}{1+y^2}.$$

(d) Give the pdf of Y when $\alpha = \beta = 1$.

Solution: We have

$$f_Y(y) = \frac{1}{\pi} \frac{1}{1+y^2}.$$

4. Consider the pdf $f_Y(y) = \pi^{-1}(1+y^2)^{-1}$ for $y \in \mathbb{R}$.

(a) Find the cdf F_Y corresponding to the pdf f_Y.

Solution: We have

$$F_Y(y) = \int_{-\infty}^{y} \frac{1}{\pi} \frac{1}{1+t^2} dt = \frac{1}{\pi} \tan^{-1}(y) + \frac{1}{2}.$$

(b) Find the transformation $g : (0,1) \to \mathbb{R}$ such that the random variable $g(U)$ has density f_Y, where $U \sim \text{Uniform}(0,1)$.

Solution: The transformation is $g(U) = \tan(\pi(U - 1/2))$, which is the inverse of the cdf.

(c) Let $Y_1, \ldots, Y_n$ be a random sample from the distribution with density f_Y. Find the pdf of the kth order statistic $Y_{(k)}$ of $Y_1, \ldots, Y_n$.

Solution: We have

$$f_{Y_{(k)}}(y) = \frac{n!}{(k-1)!(n-k)!} \left[\frac{1}{\pi} \tan^{-1}(y) + \frac{1}{2}\right]^{k-1} \left[1 - \left(\frac{1}{\pi} \tan^{-1}(y) + \frac{1}{2}\right)\right]^{n-k} \frac{1}{\pi} \frac{1}{1+y^2},$$

which we can write (cf. Question 3) as

$$f_{Y_{(k)}}(y) = \frac{\Gamma(k+(n-k+1))}{\Gamma(k)\Gamma(n-k+1)} \left[\frac{1}{\pi}\tan^{-1}(y)+\frac{1}{2}\right]^{k-1} \left[1-\left(\frac{1}{\pi}\tan^{-1}(y)+\frac{1}{2}\right)\right]^{(n-k+1)-1} \frac{1}{\pi}\frac{1}{1+y^2}.$$

Interestingly, this means that the kth order statistic of a Cauchy random sample of size n has the same distribution as the random variable $\tan(\pi(U-1/2))$, where $U \sim \text{Beta}(k, n-k+1)$.

5. Let $X_1 \sim \text{Exponential}(\lambda_1)$ and $X_2 \sim \text{Exponential}(\lambda_2)$ be independent rvs for some $\lambda_1, \lambda_2 > 0$. Define new random variables $Y_1 = X_1/X_2$ and $Y_2 = X_2$.

(a) Write down the joint pdf of X_1 and X_2.

Solution: We have

$$f_{X_1,X_1}(x_1,x_2) = \frac{1}{\lambda_1}e^{-x_1/\lambda_1}\frac{1}{\lambda_2}e^{-x_2/\lambda_2} \cdot \mathbf{1}(x_1>0, x_2>0).$$

(b) Give the joint support of the rv pair (Y_1, Y_2).

Solution: The rv pair (Y_1, Y_2) can take any values in the set $\mathcal{Y} = (0,\infty) \times (0,\infty)$.

(c) Give the Jacobian of the transformation from (X_1, X_2) to (Y_1, Y_2).

Solution:
We have

$$\begin{array}{l} y_1 = x_1/x_2 =: g_1(x_1,x_2) \\ y_2 = x_2 =: g_2(x_1,x_2) \end{array} \iff \begin{array}{l} x_1 = y_1y_2 =: g_1^{-1}(y_1,y_2) \\ x_2 = y_2 =: g_2^{-1}(y_1,y_2), \end{array}$$

so the Jacobian is

$$J(x,y) = \begin{vmatrix} \frac{\partial}{\partial y_1}y_1y_2 & \frac{\partial}{\partial y_2}y_1y_2 \\ \frac{\partial}{\partial y_1}y_2 & \frac{\partial}{\partial y_2}y_2 \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2.$$

(d) Give the joint pdf of Y_1 and Y_2.

Solution: For $(y_1, y_2) \in (0,\infty)\times(0,\infty)$ we have

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{\lambda_1}e^{-y_1y_2/\lambda_1}\frac{1}{\lambda_2}e^{-y_2/\lambda_2}|y_2| = \frac{1}{\lambda_1\lambda_2}y_2\exp\left[-y_2\left(\frac{y_1}{\lambda_1}+\frac{1}{\lambda_2}\right)\right].$$

(e) Find the cdf of Y_1 by evaluating $P(Y_1 \le y_1) = P(X_1/X_2 \le y_1)$ as a double integral.

Solution: For $y_1 > 0$, we have

$$\begin{aligned}
F_{Y_1}(y_1) &= P(Y_1 \leq y_1) \\
&= P(X_1/X_2 \leq y_1) \\
&= P(X_2 \geq X_1/y_1) \\
&= \int_0^\infty \int_{x_1/y}^\infty \frac{1}{\lambda_1} e^{-x_1/\lambda_1} \frac{1}{\lambda_2} e^{-x_2/\lambda_2} dx_2 dx_1 \\
&= \int_0^\infty \frac{1}{\lambda_1} e^{-x_1/\lambda_1} e^{-x_1/(y_1\lambda_2)} dx_2 dx_1 \\
&= \frac{1}{\lambda_1} \int_0^\infty \exp\left[-\frac{x_1}{\left(\frac{1}{\lambda_1} + \frac{1}{y_1\lambda_2}\right)^{-1}} \right] dx_1 \\
&= \frac{1}{\lambda_1} \left(\frac{1}{\lambda_1} + \frac{1}{y_1\lambda_2} \right)^{-1} \\
&= \frac{y_1}{y_1 + \lambda_1/\lambda_2}.
\end{aligned}$$

So we have

$$F_{Y1}(y_1) = \begin{cases} \frac{y_1}{y_1 + \lambda_1/\lambda_2}, & y > 0 \\ 0, & y \leq 0. \end{cases}$$

(f) Give the pdf of Y_1.

Solution: We can either integrate the joint pdf of Y_1 and Y_2 obtained in the answer to part (d) with respect to y_2 or we can take the derivative with respect to y_1 of the cdf of Y_1 obtained in the answer to part (e). Both approaches give us

$$f_{Y_1}(y_1) = \frac{\lambda_1/\lambda_2}{(y_1 + \lambda_1/\lambda_2)^2} \cdot \mathbf{1}(y_1 > 0).$$

6. Let $Z_1, \ldots, Z_n \overset{\text{ind}}{\sim} \text{Normal}(0, 1/n)$ and let $h_1, \ldots, h_n \in \mathbb{R}$. Find the distribution of $S_n = \sum_{i=1}^n h_i \cdot Z_i$.

Solution: Useing moment generating functions, we have

$$\begin{aligned} M_Y(t) &= M_{\sum_{i=1}^n h_i \cdot Z_i}(t) \\ &= \prod_{i=1}^n M_{h_i \cdot Z_i}(t) \\ &= \prod_{i=1}^n M_{Z_i}(h_i \cdot t) \\ &= \prod_{i=1}^n \exp\left[(1/n)(h_i \cdot t)^2/2\right] \\ &= \exp\left[\frac{\sum_{i=1}^n h_i^2}{n} \cdot \frac{t^2}{2}\right], \end{aligned}$$

which we recognize as the mgf of the Normal$(0, n^{-1}\sum_{i=1}^n h_i^2)$ distribution. So

$$S_n \sim \text{Normal}\left(0, n^{-1}\sum_{i=1}^n h_i^2\right).$$

7. Let $X_1 \sim$ Binomial(n_1, p) and $X_2 \sim$ Binomial(n_2, p) be independent rvs. Let $Y = X_1 + X_2$. Give the cdf $F_Y(y)$ of Y.

Solution: We can find the distribution of of Y using moment generating functions. We have

$$\begin{aligned} M_Y(t) &= M_{X_1+X_2}(t) \\ &= M_{X_1}(t) M_{X_2}(t) \\ &= [pe^t + (1-p)]^{n_1} \cdot [pe^t + (1-p)]^{n_2} \\ &= [pe^t + (1-p)]^{n_1+n_2}, \end{aligned}$$

which is the mgf of the Binomial$(n_1 + n_2, p)$ distribution. So $Y \sim$ Binomial$(n_1 + n_2, p)$ and has cdf given by

$$F_Y(y) = \begin{cases} \sum_{t \le y} \binom{n_1+n_2}{t} p^y (1-p)^{n_1+n_2-t}, & y \ge 0 \\ 0, & y < 0 \end{cases}$$