STAT 512 su 2021 Exam II

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This is a take-home test. Do not communicate with classmates about the exam until after its due date/time. You may

- *Use your notes and the lecture notes.*
- *Use books.*
- *NOT work together with others.*

Write all answers on blank sheets of paper; then take pictures and merge to a PDF. Upload a single PDF to Blackboard.

1. Copy down this sentence on your answer sheet and put your signature underneath: *I have not collaborated with any other student on this exam. The work I have presented is my own.*

2. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim}$ Uniform$(-\theta, \theta)$ for some $\theta \in (0, \infty)$.

   (a) Give the cdf of the Uniform$(-\theta, \theta)$ distribution.

   (b) Give the pdf of $X_{(n)}$.

   (c) Let $\check{\theta} = X_{(n)}$.

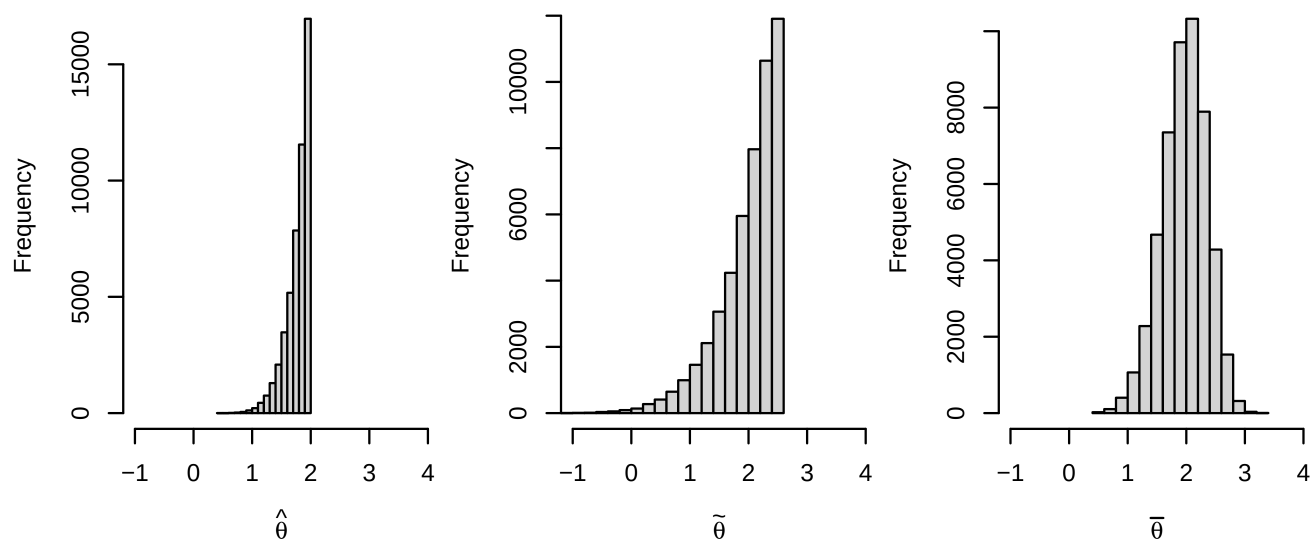
      i. Give $\operatorname{Bias} \check{\theta}$.

      ii. Suggest an estimator $\check{\theta}_{\text{unbiased}}$ based on $\check{\theta}$ which is unbiased.

   (d) Consider the three estimators of θ given by

$$\hat{\theta} = \max\{-X_{(1)}, X_{(n)}\}$$
$$\tilde{\theta} = X_{(1)} \cdot (n+1)/(1-n)$$
$$\bar{\theta} = \sqrt{3} \cdot S_n,$$

   where $S_n^2 = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$. The histograms below show 50,000 realizations of the estimators $\hat{\theta}$, $\tilde{\theta}$, and $\bar{\theta}$, respectively, under $n = 8$ and $\theta = 2$.



      i. Which estimators among $\hat{\theta}$, $\tilde{\theta}$, and $\bar{\theta}$ appear to have zero or negligible bias?

      ii. Which estimator among $\hat{\theta}$, $\tilde{\theta}$, and $\bar{\theta}$ appears to have the smallest variance?

      iii. Which estimator among $\hat{\theta}$, $\tilde{\theta}$, and $\bar{\theta}$ do you suppose has the smallest MSE?

      iv. Which estimator do you think your professor likes best?

   (e) Let $\tau = \theta^2$. Give an unbiased estimator of τ. *Hint: Use the fact that the variance of the Uniform$(-\theta, \theta)$ distribution is equal to $\theta^2/3$.*

3. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Poisson}(\lambda)$. A researcher has in mind some value $\lambda_0 > 0$, which he believes to be a likely value for the parameter λ, so he sets up an estimator of λ as

$$\hat{\lambda} = \bar{X}_n \left(\frac{n\lambda_0}{1 + n\lambda_0} \right) + \lambda_0 \left(\frac{1}{1 + n\lambda_0} \right),$$

with a view to balancing his beliefs about λ with the observed data mean $\bar{X}_n$. Note that if the researcher collects no data, that is if $n = 0$, he estimates λ with $\hat{\lambda} = \lambda_0$. If he collects a very large sample, $\hat{\lambda}$ will be close to $\bar{X}_n$, so that the data mean will feature more strongly in the estimate than his beliefs. Anyway:

(a) Give the bias of the estimator $\hat{\lambda}$.

(b) Give the variance of the estimator $\hat{\lambda}$.

(c) Determine whether $\hat{\lambda}$ is a consistent estimator of λ.

4. Let $Z_1, Z_2, Z_3, Z_4, Z_5 \overset{\text{ind}}{\sim} \text{Normal}(0, 1)$. Find the value of x in each of the following. Make use of the R functions related to the Normal, chi-squared, t, and F distributions.

(a) $P(Z_1^2 + Z_2^2 > 1) = x$.

(b) $P((Z_1 + Z_2 + Z_3)/\sqrt{3} > x) = 0.025$.

(c) $P((3/2) \cdot (Z_1^2 + Z_2^2)/(Z_3^2 + Z_4^2 + Z_5^2) < 5) = x$.

(d) $P((2/3) \cdot (Z_1^2 + Z_2^2 + Z_3^2)/(Z_4^2 + Z_5^2) < x) = 0.05$.

(e) $P(Z_3/\sqrt{(Z_1^2 + Z_2^2)/2} > 3) = x$.

(f) $P((1/\sqrt{3}) \cdot (Z_1 + Z_2 + Z_3)/\sqrt{(Z_4^2 + Z_5^2)/2} < x) = 0.9$.

5. Let $X_1, \dots, X_n$ be a random sample from the distribution with pdf given by

$$f_X(x) = \frac{1}{\gamma} x e^{-x/\sqrt{\gamma}} \cdot \mathbf{1}(x > 0)$$

for some $\gamma > 0$.

(a) Give the value to which $\bar{X}_n$ converges in probability. *Hint: f_X is the pdf of a certain Gamma distribution.*

(b) Consider the estimator $\hat{\gamma} = \bar{X}_n^2/4$ of the parameter γ.

i. Find $\text{Bias}\,\hat{\gamma}$.

ii. Give an argument that $\hat{\gamma}$ is a consistent estimator of γ. *Hint: It is difficult to compute* $\text{Var}\,\hat{\gamma}$ *and $P(|\hat{\gamma} - \gamma| < \varepsilon)$ does not admit a simple expression.*