STAT 512 su 2021 Exam II

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This is a take-home test. Do not communicate with classmates about the exam until after its due date/time. You may

- *Use your notes and the lecture notes.*
- *Use books.*
- *NOT work together with others.*

Write all answers on blank sheets of paper; then take pictures and merge to a PDF. Upload a single PDF to Blackboard.

1. Copy down this sentence on your answer sheet and put your signature underneath: *I have not collaborated with any other student on this exam. The work I have presented is my own.*

2. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(-\theta, \theta)$ for some $\theta \in (0, \infty)$.

   (a) Give the cdf of the Uniform$(-\theta, \theta)$ distribution.

   Solution: The cdf is given by

$$F_X(x) = \begin{cases} 0, & x \leq -\theta \\ \dfrac{x+\theta}{2\theta}, & -\theta < x < \theta \\ 1, & x \geq \theta. \end{cases}$$

   (b) Give the pdf of $X_{(n)}$.

   Solution: We obtain

$$f_{X_{(n)}}(x) = n \left[\frac{x+\theta}{2\theta}\right]^{n-1} \frac{1}{2\theta} \cdot \mathbf{1}(-\theta < x < \theta).$$

   (c) Let $\check{\theta} = X_{(n)}$.

   i. Give Bias $\check{\theta}$.

   Solution: We have

$$\begin{aligned} \mathbb{E}\check{\theta} &= \mathbb{E}X_{(n)} \\ &= \int_{-\theta}^{\theta} xn \left[\frac{x+\theta}{2\theta}\right]^{n-1} \frac{1}{2\theta} dx \\ &= n \int_0^1 (2\theta u - \theta) u^{n-1} du \qquad \text{by setting } u = (x+\theta)/(2\theta) \\ &\quad \vdots \\ &= \frac{n-1}{n+1} \cdot \theta. \end{aligned}$$

   So the bias of $\check{\theta}$ is given by

$$\text{Bias}\,\check{\theta} = \mathbb{E}\check{\theta} - \theta = \frac{n-1}{n+1} \cdot \theta - \theta = -\theta \cdot \frac{2}{n+1}.$$

   ii. Suggest an estimator $\check{\theta}_{\text{unbiased}}$ based on $\check{\theta}$ which is unbiased.

Solution: The estimator

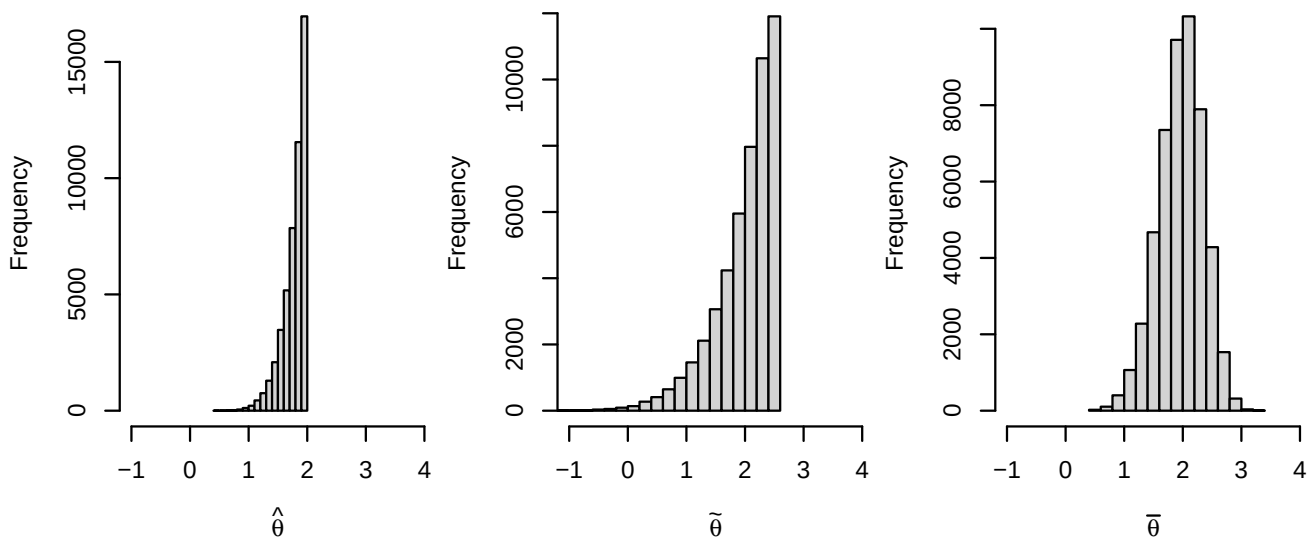
$$\check{\theta}_{\text{unbiased}} = \frac{n+1}{n-1} \cdot \check{\theta}$$

is an unbiased estimator of θ.

(d) Consider the three estimators of θ given by

$$\begin{aligned}
\hat{\theta} &= \max\{-X_{(1)}, X_{(n)}\} \\
\tilde{\theta} &= X_{(1)} \cdot (n+1)/(1-n) \\
\bar{\theta} &= \sqrt{3} \cdot S_n,
\end{aligned}$$

where $S_n^2 = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$. The histograms below show 50,000 realizations of the estimators $\hat{\theta}$, $\tilde{\theta}$, and $\bar{\theta}$, respectively, under $n = 8$ and $\theta = 2$.



i. Which estimators among $\hat{\theta}$, $\tilde{\theta}$, and $\bar{\theta}$ appear to have zero or negligible bias?

Solution: The estimators $\tilde{\theta}$ and $\bar{\theta}$ appear to have zero or negligible bias, as their histograms are centered around the true value of the parameter, $\theta = 2$. That is, the "balancing points" of these histograms are at the value $\theta = 2$. In fact, the estimator $\tilde{\theta}$ is unbiased, while the estimator $\bar{\theta}$ has a small bias, though the small bias is not apparent in these plots.

ii. Which estimator among $\hat{\theta}$, $\tilde{\theta}$, and $\bar{\theta}$ appears to have the smallest variance?

Solution: The estimator $\hat{\theta}$ appears to have the smallest variance, as its histogram has the smallest spread.

iii. Which estimator among $\hat{\theta}$, $\tilde{\theta}$, and $\bar{\theta}$ do you suppose has the smallest MSE?

Solution: The estimator $\hat{\theta}$ looks like it should have the smallest MSE (indeed it does), as its variance is much less than that of the other two estimators, and its bias is rather small.

iv. Which estimator do you think your professor likes best?

Solution: Your professor cares more about having a small MSE than about having an unbiased estimator—a preference he may or may not have successfully imparted to his class!—so he prefers the estimator $\hat{\theta}$ to the other two, even though it exhibits a greater bias.

(e) Let $\tau = \theta^2$. Give an unbiased estimator of τ. *Hint: Use the fact that the variance of the Uniform*$(-\theta, \theta)$ *distribution is equal to* $\theta^2/3$.

Solution: We know that the sample variance S_n^2 is an unbiased estimator of the population variance as long as the latter is finite. Therefore $\mathbb{E}S_n^2 = \theta^2/3$, giving that

$$\hat{\tau} = 3 \cdot S_n^2$$

is an unbiased estimator of $\tau = \theta^2$.

3. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Poisson}(\lambda)$. A researcher has in mind some value $\lambda_0 > 0$, which he believes to be a likely value for the parameter λ, so he sets up an estimator of λ as

$$\hat{\lambda} = \bar{X}_n \left(\frac{n\lambda_0}{1 + n\lambda_0} \right) + \lambda_0 \left(\frac{1}{1 + n\lambda_0} \right),$$

with a view to balancing his beliefs about λ with the observed data mean $\bar{X}_n$. Note that if the researcher collects no data, that is if $n = 0$, he estimates λ with $\hat{\lambda} = \lambda_0$. If he collects a very large sample, $\hat{\lambda}$ will be close to $\bar{X}_n$, so that the data mean will feature more strongly in the estimate than his beliefs. Anyway:

(a) Give the bias of the estimator $\hat{\lambda}$.

Solution: Using the fact that $\mathbb{E}\bar{X}_n = \lambda$, We have

$$\text{Bias}\,\hat{\lambda} = (\lambda_0 - \lambda) \cdot \frac{1}{1 + n\lambda_0},$$

(b) Give the variance of the estimator $\hat{\lambda}$.

Solution: Using the fact that $\text{Var}\,\bar{X}_n = \lambda/n$, we have

$$\text{Var}\,\hat{\lambda} = \left(\frac{n\lambda_0}{1 + n\lambda_0} \right)^2 \frac{\lambda}{n}.$$

(c) Determine whether $\hat{\lambda}$ is a consistent estimator of λ.

Solution: The estimator $\hat{\lambda}$ is consistent for λ if $\text{MSE}\,\hat{\lambda}$ goes to zero as $n \to \infty$. This will be the case if $\text{Bias}\,\hat{\lambda}$ and $\text{Var}\,\hat{\lambda}$ both go to zero as $n \to \infty$. We see that

$$\lim_{n\to\infty} \text{Bias}\,\hat{\lambda} = \lim_{n\to\infty} (\lambda_0 - \lambda) \cdot \frac{1}{1 + n\lambda_0} = 0$$

and

$$\lim_{n\to\infty} \text{Var}\,\hat{\lambda} = \lim_{n\to\infty} \left(\frac{n\lambda_0}{1 + n\lambda_0} \right)^2 \frac{\lambda}{n} = 0.$$

Therefore $\hat{\lambda}$ is a consistent estimator for λ.

4. Let $Z_1, Z_2, Z_3, Z_4, Z_5 \overset{\text{ind}}{\sim} \text{Normal}(0, 1)$. Find the value of x in each of the following. Make use of the R functions related to the Normal, chi-squared, t, and F distributions.

(a) $P(Z_1^2 + Z_2^2 > 1) = x$.

Solution: We have $Z_1^2 + Z_2^2 \sim \chi_2^2$, so $x =$ `1 - pchisq(1,2)` $= 0.6065307$.

(b) $P((Z_1 + Z_2 + Z_3)/\sqrt{3} > x) = 0.025$.

Solution: We have $(Z_1 + Z_2 + Z_3)/\sqrt{3} \sim \text{Normal}(0, 1)$, so $x =$ `qnorm(0.975)` $= 1.959964$.

(c) $P((3/2) \cdot (Z_1^2 + Z_2^2)/(Z_3^2 + Z_4^2 + Z_5^2) < 5) = x$.

Solution: We have $(3/2) \cdot (Z_1^2 + Z_2^2)/(Z_3^2 + Z_4^2 + Z_5^2) \sim F_{2,3}$, so $x =$ `pf(5,2,3)` $= 0.889142$.

(d) $P((2/3) \cdot (Z_1^2 + Z_2^2 + Z_3^2)/(Z_4^2 + Z_5^2) < x) = 0.05$.

Solution: We have $(2/3) \cdot (Z_1^2 + Z_2^2 + Z_3^2)/(Z_4^2 + Z_5^2) \sim F_{3,2}$, so $x =$ `qf(0.05,3,2)` $= 0.1046891$.

(e) $P(Z_3/\sqrt{(Z_1^2 + Z_2^2)/2} > 3) = x$.

Solution: We have $Z_3/\sqrt{(Z_1^2 + Z_2^2)/2} \sim t_2$ so $x =$ `1 - pt(3,2)` $= 0.04773298$

(f) $P((1/\sqrt{3}) \cdot (Z_1 + Z_2 + Z_3)/\sqrt{(Z_4^2 + Z_5^2)/2} < x) = 0.9$.

Solution: We have $(1/\sqrt{3}) \cdot (Z_1 + Z_2 + Z_3)/\sqrt{(Z_4^2 + Z_5^2)/2} \sim t_2$, so $x =$ `qt(0.9,2)` $= 1.885618$.

5. Let $X_1, \ldots, X_n$ be a random sample from the distribution with pdf given by

$$f_X(x) = \frac{1}{\gamma} x e^{-x/\sqrt{\gamma}} \cdot \mathbf{1}(x > 0)$$

for some $\gamma > 0$.

(a) Give the value to which $\bar{X}_n$ converges in probability. *Hint: f_X is the pdf of a certain Gamma distribution.*

Solution: We have $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Gamma}(2, \sqrt{\gamma})$, so the population mean is $2\sqrt{\gamma}$. By the WLLN, $\bar{X}_n \xrightarrow{p} 2\sqrt{\gamma}$.

(b) Consider the estimator $\hat{\gamma} = \bar{X}_n^2/4$ of the parameter γ.

i. Find $\text{Bias}\,\hat{\gamma}$.

Solution: We have $\mathbb{E}\bar{X}_n^2 = \text{Var}\,\bar{X}_n + (\mathbb{E}\bar{X}_n)^2 = 2(\sqrt{\gamma})^2/n + (2\sqrt{\gamma})^2$, so

$$\mathbb{E}\hat{\gamma} = \gamma + \frac{\gamma}{2n} \qquad \Longrightarrow \qquad \text{Bias}\,\hat{\gamma} = \frac{\gamma}{2n}.$$

ii. Give an argument that $\hat{\gamma}$ is a consistent estimator of γ. *Hint: It is difficult to compute* $\text{Var}\,\hat{\gamma}$ *and* $P(|\hat{\gamma} - \gamma| < \varepsilon)$ *does not admit a simple expression.*

Solution: Since $\tau(x) = x^2/4$ is a continuous function, we can argue that since $\bar{X}_n \xrightarrow{p} 2\sqrt{\gamma}$, it follows that $\bar{X}_n^2/4 \xrightarrow{p} (2\sqrt{\gamma})^2/4 = \gamma$.