STAT 513 fa 2020 Lec 02

Inference about the mean and variance of a Normal population

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Inference about the mean and variance of a Normal population

- Here we consider the case in which $X_1, \ldots, X_n$ is a random sample from the $\text{Normal}(\mu, \sigma^2)$ distribution, where $\mu \in (-\infty, \infty)$ and $\sigma^2 \in (0, \infty)$ are unknown.

- We wish to make inference, i.e. test hypotheses about μ and σ^2 based on $X_1, \ldots, X_n$.

- The following pivot quantity results (from STAT 512) will play a crucial role:

  If $X_1, \ldots, X_n$ is a random sample from the $\text{Normal}(\mu, \sigma^2)$ distribution, then

  1. $\sqrt{n}(\bar{X}_n - \mu)/\sigma \sim \text{Normal}(0, 1)$
  2. $\sqrt{n}(\bar{X}_n - \mu)/S_n \sim t_{n-1}$
  3. $(n-1)S_n^2/\sigma^2 \sim \chi^2_{n-1}$

Inference about the mean of a Normal population

Assuming known variance

- At first we will assume that σ^2 is known, so that only μ is unknown.

- For some value μ_0, which we will call the null value of μ, we consider for some constants C_1 and C_2 the following tests of hypotheses:

  1. *Right-tailed test:* Test H_0: $\mu \leq \mu_0$ versus H_1: $\mu > \mu_0$ with the test

  $$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} > C_1.$$

  2. *Left-tailed test:* Test H_0: $\mu \geq \mu_0$ versus H_1: $\mu < \mu_0$ with the test

  $$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} < -C_1.$$

3. *Two-sided test:* Test H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$ with the test

$$\text{Reject } H_0 \text{ iff } \left| \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} \right| > C_2.$$

- These tests are sometimes called Z-tests, which comes from the fact that if $X_1, \ldots, X_n$ is a random sample from the Normal(μ, σ^2) distribution, then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0, 1)$$

  and Normal$(0, 1)$ random variables are often represented with Z.

- For $\xi \in (0, 1)$, let z_ξ be the value which satisfies $\xi = P(Z > z_\xi)$, when $Z \sim \text{Normal}(0, 1)$.

- Let Φ denote the cdf of the Normal$(0, 1)$ distribution.

- The rejection regions for these tests are defined by the values C_1 and C_2. We refer to such values as *critical values*.

- We may choose or calibrate the critical values C_1 and C_2 such that the tests have a desired size.

- Each test is based on how far $\bar{X}_n$ is away from from μ_0 in terms of how many standard errors. The standard error of $\bar{X}_n$ is $\sigma/\sqrt{n}$.

- The first two tests are called *one-sided* tests. The right-tailed test rejects H_0 if $\bar{X}_n$ lies more than C_1 standard errors above (to the right of) μ_0 and left-tailed test rejects H_0 if $\bar{X}_n$ lies more than C_1 standard errors below (to the left of) μ_0.

- The two-sided test rejects H_0 when $\bar{X}_n$ lies more than C_2 standard errors from μ_0 in either direction.

- **Exercise:** *Suppose $X_1, \ldots, X_n$ is a random sample from the Normal$(\mu, 1/10)$ distribution, where μ is unknown. Researchers wish to test H_0: $\mu = 5$ versus H_1: $\mu \neq 5$ using two-sided test from above.*

  (i) *Suppose $\mu = 5$. What is the probability of a Type I error if the critical value $C_2 = 3$ is used?*

  (ii) *Suppose you wish to control the Type I error probability at $\alpha = 0.01$. What value of C_2 should you use?*

  (iii) *Using this value of C_2, compute the probability of rejecting H_0 when $\mu = 5.4$ if the sample size is $n = 9$.*

  (iv) *What is the effect on this probability if a larger value of C_2 used? What is the effect on the size of the test?*

Answers:

(i) The Type I error probability is

$$P_{\mu=5}(|\sqrt{n}(\bar{X}_n - 5)/(1/\sqrt{10})| > 3) = P(|Z| > 3), \quad Z \sim \text{Normal}(0,1)$$
$$= 2P(Z > 3)$$
$$= 2[1 - \Phi(3)]$$
$$= \texttt{2*(1 - pnorm(3))}$$
$$= 0.002699796.$$

(ii) The Type I error probability for a given critical value C_2 is, from the above,

$$P_{\mu=5}(|\sqrt{n}(\bar{X}_n - 5)/(1/\sqrt{10})| > C_2) = \cdots = 2[1 - \Phi(C_2)].$$

Setting this equal to 0.01 gives $C_2 = 2.576$, since

$$2[1 - \Phi(C_2)] = 0.01 \iff .995 = \Phi(C_2) \iff C_2 = z_{.005} = \texttt{qnorm(.995)} = 2.575829.$$

(iii) With $C_2 = 2.576$, if $\mu = 5.4$, the probability of rejecting H_0 is

$$P_{\mu=5.4}(|\sqrt{9}(\bar{X}_9 - 5)/(1/\sqrt{10})| > 2.576)$$
$$= 1 - P_{\mu=5.4}(-2.576 < \sqrt{9}(\bar{X}_9 - 5)/(1/\sqrt{10}) < 2.576)$$
$$= 1 - P_{\mu=5.4}(-2.576 < \sqrt{9}(\bar{X}_9 - 5.4)/(1/\sqrt{10}) + \sqrt{9}(5.4 - 5)/(1/\sqrt{10}) < 2.576)$$
$$= 1 - P_{\mu=5.4}(-2.576 < Z + \sqrt{9}(5.4 - 5)/(1/\sqrt{10}) < 2.576), \quad Z \sim \text{Normal}(0,1)$$
$$= 1 - P_{\mu=5.4}(-2.576 - \sqrt{9}(5.4 - 5)/(1/\sqrt{10}) < Z < 2.576 - \sqrt{9}(5.4 - 5)/(1/\sqrt{10}))$$
$$= 1 - [P(Z < 2.576 - \sqrt{9}(5.4 - 5)/(1/\sqrt{10})) - P(Z < -2.576 - \sqrt{9}(5.4 - 5)/(1/\sqrt{10}))]$$
$$= 1 - [\Phi(2.576 - \sqrt{90}(5.4 - 5)) - \Phi(-2.576 - \sqrt{90}(5.4 - 5))]$$
$$= \texttt{1-(pnorm(2.576-sqrt(90)*(5.4-5))-pnorm(-2.576-sqrt(90)*(5.4-5)))}$$
$$= 0.8885273.$$

(iv) If a larger critical value C_2 were used, stronger evidence would be required to reject H_0. So this probability would decrease. The size would decrease too, because under the null hypothesis of $\mu = 5$, the probability of obtaining a random sample for which test statistic exceeded C_2 would be smaller.

- **Exercise:** *Suppose $X_1, \ldots, X_n$ is a random sample from the Normal$(\mu, 3)$ distribution, where μ is unknown. Researchers wish to test H_0: $\mu \leq 2$ versus H_1: $\mu > 2$ using the right-tailed test from above with $C_1 = 2$.*

  (i) *Suppose the true mean is $\mu = 3$. If the sample size is $n = 5$, what is the probability that the test will reject H_0? What is the effect on this probability of increasing the sample size?*

3

(ii) *Plot the power curve of the test for the sample sizes $n = 5, 10, 20$. Compute the size of the test and add a horizontal line to the plot with height equal to the size.*

(iii) *Is the size of the test affected by the sample size n?*

Answers:

(i) Compute

$$\begin{aligned}
P_{\mu=3}(\sqrt{5}(\bar{X}_5 - 2)/\sqrt{3} > 2) &= P_\mu(\sqrt{5}(\bar{X}_5 - 3)/\sqrt{3} + \sqrt{5}(3 - 2)/\sqrt{3} > 2) \\
&= P(Z > 2 - \sqrt{5}(3 - 2)/\sqrt{3}), \quad Z \sim \text{Normal}(0, 1) \\
&= 1 - \Phi(2 - \sqrt{5}/\sqrt{3}) \\
&= \texttt{1-pnorm(2 - sqrt(5)/sqrt(3))} \\
&= 0.2391605.
\end{aligned}$$

Increasing n will increase the probability of rejecting H_0 when $\mu = 3$.

(ii) For any $\mu \in (-\infty, \infty)$, the power under a given sample size n is given by

$$\begin{aligned}
\gamma(\mu) &= P_\mu(\sqrt{n}(\bar{X}_n - 2)/\sqrt{3} > 2) \\
&= P_\mu(\sqrt{n}(\bar{X}_n - \mu)/\sqrt{3} + \sqrt{n}(\mu - 2)/\sqrt{3} > 2) \\
&= P(Z > 2 - \sqrt{n}(\mu - 2)/\sqrt{3}), \quad Z \sim \text{Normal}(0, 1) \\
&= 1 - \Phi(2 - \sqrt{n}(\mu - 2)/\sqrt{3}).
\end{aligned}$$

The size of the test is

$$\sup_{\mu \le 2} \gamma(\mu) = \gamma(2) = 1 - \Phi(2) = \texttt{1 - pnorm(2)} = 0.02275013.$$

The following R code makes the plot:

```
mu.seq <- seq(1,4,length=100)
mu.0 <- 2

power.n5 <- 1-pnorm(2-sqrt(5)*(mu.seq - mu.0)/sqrt(3))
power.n10 <- 1-pnorm(2-sqrt(10)*(mu.seq - mu.0)/sqrt(3))
power.n20 <- 1-pnorm(2-sqrt(20)*(mu.seq - mu.0)/sqrt(3))

plot(mu.seq,power.n5,type="l",ylim=c(0,1),xlab="mu",ylab="power")
lines(mu.seq,power.n10,lty=2)
lines(mu.seq,power.n20,lty=4)
abline(v=mu.0,lty=3)      # vert line at null value
abline(h=0.02275013,lty=3) # horiz line at size
```

4
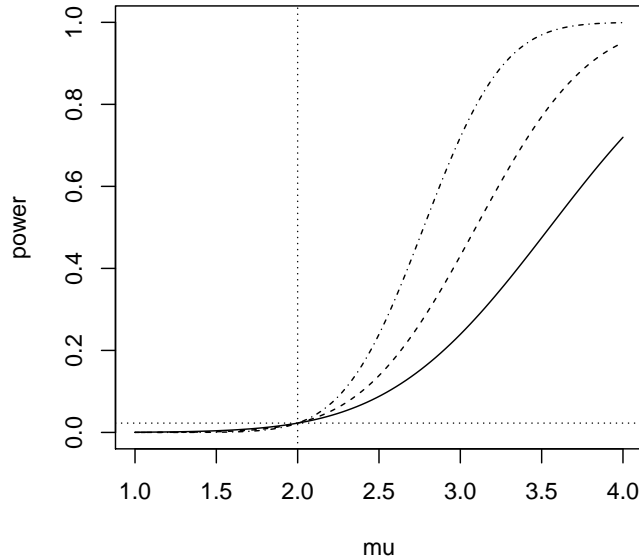
(iii) The sample size does not affect the size of the test.

- **Exercise:** *Get expressions for the power functions and the sizes of the right-tailed, left-tailed, and two-sided Z-tests. Moreover, for any $\alpha \in (0,1)$, give the values C_1 and C_2 such that the tests have size α.*

Answers:

1. The power function for the right-tailed Z test is

$$
\begin{aligned}
\gamma(\mu) &= P_\mu(\sqrt{n}(\bar{X}_n - \mu_0)/\sigma > C_1) \\
&= P_\mu(\sqrt{n}(\bar{X}_n - \mu)/\sigma + \sqrt{n}(\mu - \mu_0)/\sigma > C_1) \\
&= P_\mu(\sqrt{n}(\bar{X}_n - \mu)/\sigma > C_1 - \sqrt{n}(\mu - \mu_0)/\sigma) \\
&= P(Z > C_1 - \sqrt{n}(\mu - \mu_0)/\sigma), \quad Z \sim \mathrm{Normal}(0,1) \\
&= 1 - \Phi(C_1 - \sqrt{n}(\mu - \mu_0)/\sigma).
\end{aligned}
$$

The size is given by
$$
\sup_{\mu \leq \mu_0} \gamma(\mu) = \gamma(\mu_0) = 1 - \Phi(C_1).
$$

The test will have size equal to α if $C_1 = z_\alpha$, since

$$
\alpha = 1 - \Phi(C_1) \iff C_1 = z_\alpha.
$$

2. The power function for the left-tailed Z test is

$$
\begin{aligned}
\gamma(\mu) &= P_\mu(\sqrt{n}(\bar{X}_n - \mu_0)/\sigma < -C_1) \\
&= P_\mu(\sqrt{n}(\bar{X}_n - \mu)/\sigma + \sqrt{n}(\mu - \mu_0)/\sigma < -C_1) \\
&= P_\mu(\sqrt{n}(\bar{X}_n - \mu)/\sigma < -C_1 - \sqrt{n}(\mu - \mu_0)/\sigma) \\
&= P(Z < -C_1 - \sqrt{n}(\mu - \mu_0)/\sigma), \quad Z \sim \text{Normal}(0, 1) \\
&= \Phi(-C_1 - \sqrt{n}(\mu - \mu_0)/\sigma).
\end{aligned}
$$

The size is given by
$$
\sup_{\mu \geq \mu_0} \gamma(\mu) = \gamma(\mu_0) = \Phi(-C_1).
$$

The test will have size equal to α if $C_1 = z_\alpha$, since

$$
\alpha = \Phi(-C_1) \iff C_1 = z_\alpha.
$$

3. The power function for the two-sided Z-test is

$$
\begin{aligned}
\gamma(\mu) &= P_\mu(|\sqrt{n}(\bar{X}_n - \mu_0)/\sigma| > C_2) \\
&= 1 - P_\mu(-C_2 < \sqrt{n}(\bar{X}_n - \mu_0)/\sigma < C_2) \\
&= 1 - P_\mu(-C_2 < \sqrt{n}(\bar{X}_n - \mu)/\sigma + \sqrt{n}(\mu - \mu_0)/\sigma < C_2) \\
&= 1 - P_\mu(-C_2 - \sqrt{n}(\mu - \mu_0)/\sigma < \sqrt{n}(\bar{X}_n - \mu)/\sigma < C_2 - \sqrt{n}(\mu - \mu_0)/\sigma) \\
&= 1 - P(-C_2 - \sqrt{n}(\mu - \mu_0)/\sigma < Z < C_2 - \sqrt{n}(\mu - \mu_0)/\sigma), \quad Z \sim \text{Normal}(0, 1) \\
&= 1 - [\Phi(C_2 - \sqrt{n}(\mu - \mu_0)/\sigma) - \Phi(-C_2 - \sqrt{n}(\mu - \mu_0)/\sigma)].
\end{aligned}
$$

The size is given by

$$
\sup_{\mu \in \{\mu_0\}} \gamma(\mu) = \gamma(\mu_0) = 1 - [\Phi(C_2) - \Phi(-C_2)] = 2[1 - \Phi(C_2)].
$$

The test will have size equal to α if $C_2 = z_{\alpha/2}$, since

$$
\alpha = 2[1 - \Phi(C_2)] \iff 1 - \alpha/2 = \Phi(C_2) \iff C_2 = z_{\alpha/2}.
$$

- **Exercise:** *Let $X_1, \ldots, X_n$ be a random sample from the* Normal$(\mu, 5)$ *distribution, where μ is unknown. Researchers are interested in testing the hypotheses H_0: $\mu \geq 10$ versus H_1: $\mu < 10$.*

  (i) *Give a test that has size $\alpha = 0.05$.*

  (ii) *Reformulate the size-α test from part (a) so that it takes the form* Reject H_0 iff $\bar{X}_n < C$. *Give the value of C when $n = 10$.*

  (iii) *Draw a picture of the density of $\bar{X}_n$ under $\mu = 10$ and under $\mu = 8.5$; then shade the areas which represent the power and the size of the test. Compute the power of the test if $n = 10$.*

  Answers:

  1. Use the test Reject H_0 iff $\sqrt{n}(\bar{X}_n - 10)/\sqrt{5} < -z_{0.05}$, where $z_{0.05} = $ `qnorm(.95)` $= 1.644854$.

  2. This is equivalent to the test Reject H_0 iff $\bar{X}_n < 10 - z_{0.05}\sqrt{5}/\sqrt{n}$. For $n = 10$, the test becomes Reject H_0 iff $\bar{X}_n < 8.836913$.

  3. The left-hand panel below shows the densities of $\bar{X}_{10}$ under $\mu = 10$ and $\mu = 8.5$. The area of the red region is the size and that of the green (overlapping with the red) region is the power of the test. The shaded areas lie to the left of the value 8.836913. The right-hand panel shows the power function over many values of μ. At $\mu = 8.5$, the power is

$$P_{\mu=8.5}(\sqrt{n}(\bar{X}_n - 10)/\sqrt{5} < -z_{0.05}) = \Phi(-z_{0.05} - \sqrt{10}(8.5 - 10)/\sqrt{5})$$
$$= \texttt{pnorm(-qnorm(.95)-sqrt(10)*(8.5-10)/sqrt(5))}$$
$$= 0.683129.$$
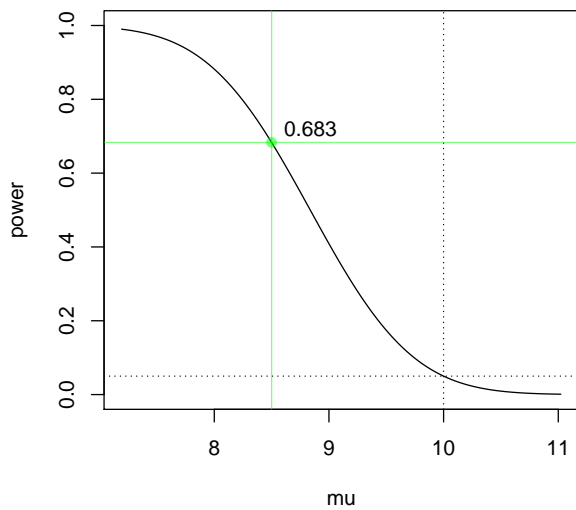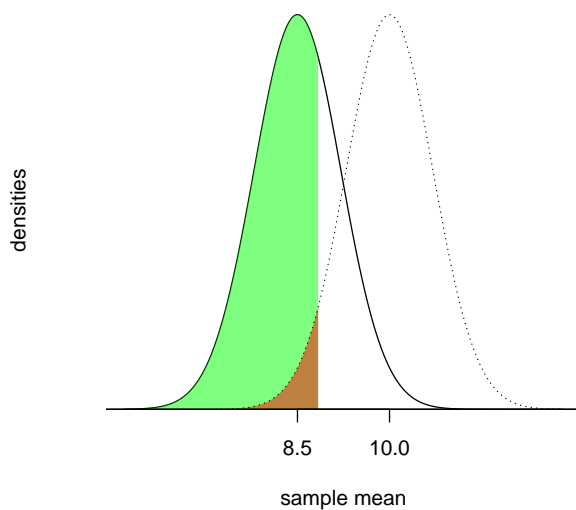
- **Formulas concerning tests about Normal mean (when variance known):** Suppose $X_1, \ldots, X_n$ is a random sample from the Normal(μ, σ^2) distribution, where μ is unknown but σ^2 is known. Then we have the following, where Φ is the cdf of the Normal$(0, 1)$ distribution and $Z_n = \sqrt{n}(\bar{X}_n - \mu_0)/\sigma$:

H_0	H_1	Reject H_0 at α iff	Power function $\gamma(\mu)$
$\mu \leq \mu_0$	$\mu > \mu_0$	$Z_n > z_\alpha$	$1 - \Phi(z_\alpha - \sqrt{n}(\mu - \mu_0)/\sigma)$.
$\mu \geq \mu_0$	$\mu < \mu_0$	$Z_n < -z_\alpha$	$\Phi(-z_\alpha - \sqrt{n}(\mu - \mu_0)/\sigma)$
$\mu = \mu_0$	$\mu \neq \mu_0$	$|Z_n| > z_{\alpha/2}$	$1 - [\Phi(z_{\alpha/2} - \sqrt{n}(\mu - \mu_0)/\sigma) - \Phi(-z_{\alpha/2} - \sqrt{n}(\mu - \mu_0)/\sigma)]$

- **Exercise:** *Let $X_1, \ldots, X_n$ be a random sample from the Normal$(\mu, 4)$ distribution, where μ is unknown, and suppose there are three researchers:*

  - *Efstathios to test H_0: $\mu \leq 5$ versus H_1: $\mu > 5$ with test Reject H_0 iff $\sqrt{n}(\bar{X}_n - 5)/2 > z_{0.10}$*
  - *Dimitris to test H_0: $\mu \geq 5$ versus H_1: $\mu < 5$ with test Reject H_0 iff $\sqrt{n}(\bar{X}_n - 5)/2 < -z_{0.10}$*
  - *Phoebe to test H_0: $\mu = 5$ versus H_1: $\mu \neq 5$ with test Reject H_0 iff $|\sqrt{n}(\bar{X}_n - 5)/2| > z_{0.05}$*

 *Each will collect a sample of size $n = 20$.*

  (i)   *If the true value of the mean is $\mu = 4.5$, which researcher may commit a Type I error?*

  (ii) *If the true value of the mean is $\mu = 5.5$, which researcher may commit a Type I error?*

  (iii) *If the true value of the mean is $\mu = 4.5$, which researchers may commit a Type II error?*

  (iv) *If the true value of the mean is $\mu = 5.5$, which researchers may commit a Type II error?*

  (v)   *What is the size of each test?*

  (vi) *If the true value of the mean is $\mu = 4.5$, which researcher is most likely to reject H_0?*

  (vii) *If the true value of the mean is $\mu = 4.5$, which researcher is least likely to reject H_0?*

  (viii) *Compute the power of each researcher's test when $\mu = 4.5$.*

  (ix) *Plot the power curves of the three researchers' tests together.*

Answers:

(viii) The following graphics illustrate the power calculation for each researcher's test and display the power in the right-hand panel:
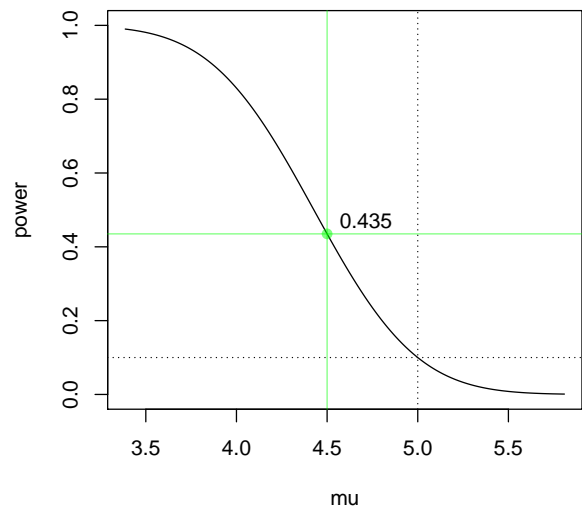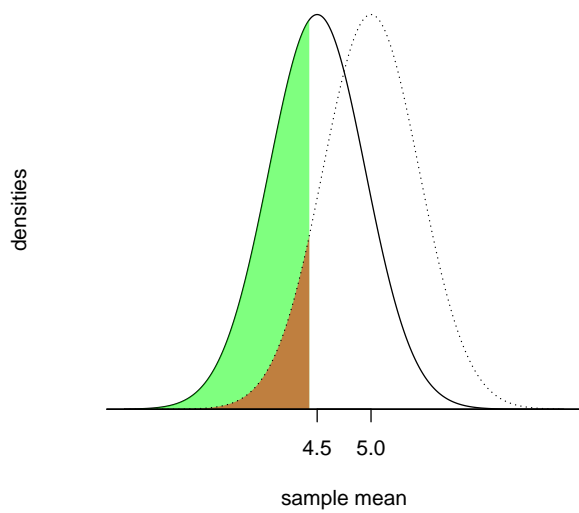
The test of Efstathios has power

$$1 - \Phi(z_{0.10} - \sqrt{20}(4.5 - 5)/2) = \texttt{1-pnorm(qnorm(.9)-sqrt(20)*(4.5-5)/2)} = 0.0082.$$
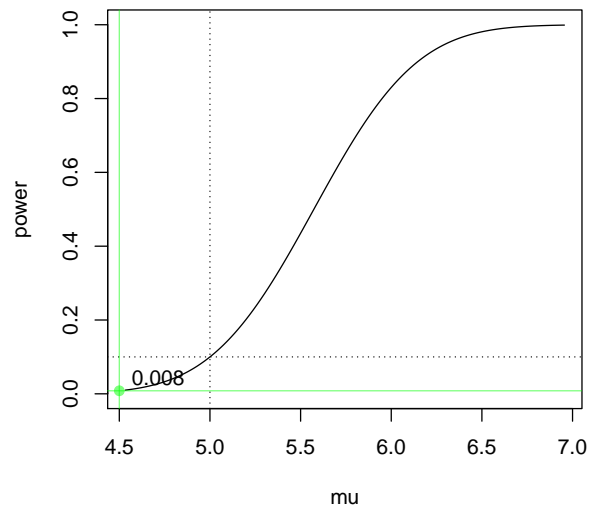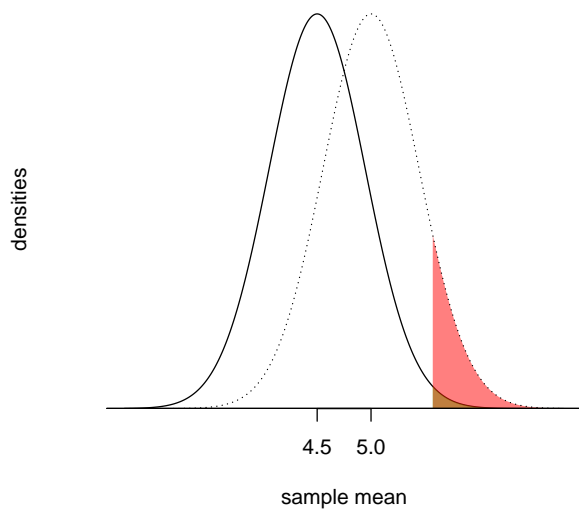
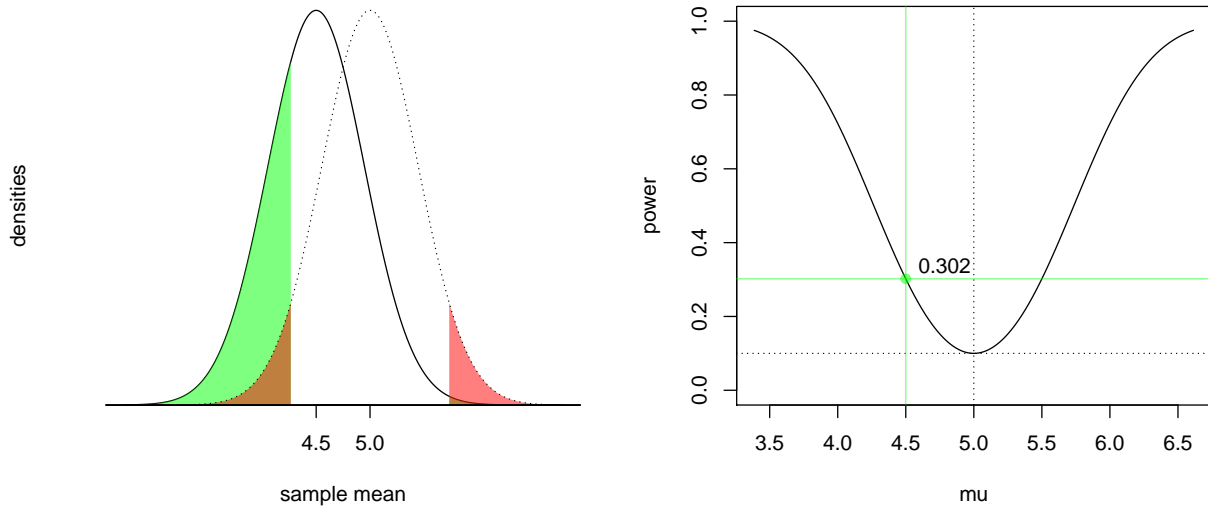The test of Dimitris has power

$$\Phi(-z_{0.10} - \sqrt{20}(4.5 - 5)/2) = \texttt{pnorm(-qnorm(.9)-sqrt(20)*(4.5-5)/2)} = 0.4351.$$

The test of Phoebe has power

$$1 - [\Phi(z_{0.05} - \sqrt{20}(4.5 - 5)/2) - \Phi(-z_{0.05} - \sqrt{20}(4.5 - 5)/2)]$$
$$= \texttt{1-(pnorm(qnorm(.95)-sqrt(20)*(4.5-5)/2)-pnorm(-qnorm(.95)-sqrt(20)*(4.5-5)/2))}$$
$$= 0.302024.$$

9

(ix) The following R code makes the plot:

```
mu.seq <- seq(3,7,length=100)
mu.0 <- 5

power.E <- 1-pnorm(qnorm(.9)-sqrt(20)*(mu.seq - mu.0)/2)
power.D <- pnorm(-qnorm(.9)-sqrt(20)*(mu.seq - mu.0)/2)
power.H <- 1-(pnorm(qnorm(.95)-sqrt(20)*(mu.seq - mu.0)/2)
             - pnorm(-qnorm(.95)-sqrt(20)*(mu.seq - mu.0)/2))

plot(mu.seq, power.E,type="l",ylim=c(0,1),xlab="mu",ylab="power")
lines(mu.seq, power.D,lty=2)
lines(mu.seq, power.H,lty=4)
abline(v=mu.0,lty=3) # vert line at null value
abline(h=0.10,lty=3) # horiz line at size
```
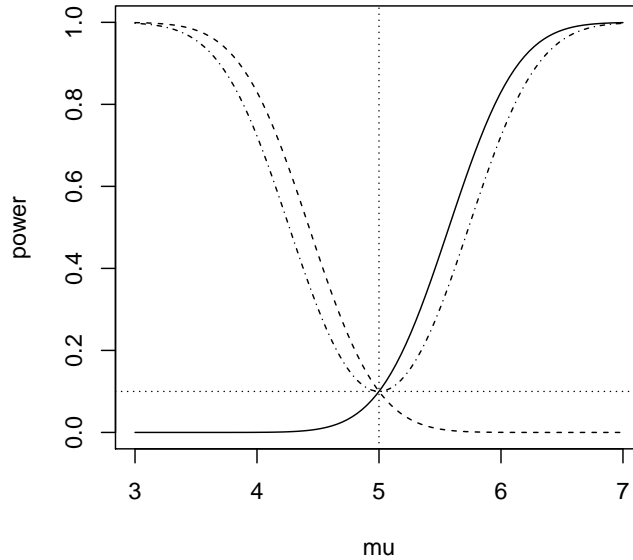
Unknown variance

- In practice σ^2 is unknown, and we estimate it with the sample variance S_n^2.

- We now consider, for some constants C_1 and C_2, the following tests of hypotheses:

  1. *Right-tailed test:* Test H_0: $\mu \leq \mu_0$ versus H_1: $\mu > \mu_0$ with the test

  $$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} > C_1.$$

  2. *Left-tailed test:* Test H_0: $\mu \geq \mu_0$ versus H_1: $\mu < \mu_0$ with the test

  $$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} < -C_1.$$

  3. *Two-sided test:* Test H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$ with the test

  $$\text{Reject } H_0 \text{ iff } \left| \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \right| > C_2.$$

- These tests are sometimes called t-tests, which comes from the fact that if $X_1, \ldots, X_n$ is a random sample from the Normal(μ, σ^2) distribution, then

$$\frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{n-1},$$

where t_{n-1} represents the t distribution with $n - 1$ degrees of freedom.

11

- For $\xi \in (0,1)$, let $t_{n-1,\xi}$ be the value which satisfies $\xi = P(T > t_{n-1,\xi})$, when $T \sim t_{n-1}$.

- Let $F_{t_{n-1}}$ denote the cdf of the t-distribution with $n-1$ degrees of freedom.

- **Exercise:** *For any $\alpha \in (0,1)$, find the values C_1 and C_2 such that the above tests have size α.*

Answer: Until now, we have computed the size by writing down the power function and taking its supremum over the parameter values in the null space. Each time we have done this, the result has been that the size is simply equal to the power at the null value (the value at the boundary of the null space or the unique value in the null space if this is single-valued). It will be the same in this case, so we will take a shortcut: We consider for each test the probability that it rejects H_0 when $\mu = \mu_0$. Then we set this equal to α and solve for the critical value.

1. For the right-tailed t-test we get

$$\begin{aligned}
\alpha &= P_{\mu=\mu_0}(\sqrt{n}(\bar{X}_n - \mu_0)/S_n > C_1) \\
&= P(T > C_1), \quad T \sim t_{n-1} \\
&= 1 - F_{t_{n-1}}(C_1),
\end{aligned}$$

which gives $C_1 = t_{n-1,\alpha}$.

2. For the left-tailed t-test we get

$$\begin{aligned}
\alpha &= P_{\mu=\mu_0}(\sqrt{n}(\bar{X}_n - \mu_0)/S_n < -C_1) \\
&= P(T < -C_1), \quad T \sim t_{n-1} \\
&= F_{t_{n-1}}(-C_1),
\end{aligned}$$

which is satisfied when $C_1 = t_{n-1,\alpha}$.

3. For the two-sided t-test we get

$$\begin{aligned}
\alpha &= P_{\mu=\mu_0}(|\sqrt{n}(\bar{X}_n - \mu_0)/S_n| < -C_2) \\
&= 1 - P_{\mu=\mu_0}(-C_2 < \sqrt{n}(\bar{X}_n - \mu_0)/S_n < C_2) \\
&= 1 - P(-C_2 < T < C_2), \quad T \sim t_{n-1} \\
&= 1 - [F_{t_{n-1}}(C_2) - F_{t_{n-1}}(-C_2)],
\end{aligned}$$

which gives $C_2 = t_{n-1,\alpha/2}$.

- **Exercise:** *Let $X_1, \ldots, X_{20}$ be a random sample from the Normal(μ, σ^2) distribution, where μ and σ are unknown.*

  (i) *Give a test of H_0: $\mu \geq -1$ versus H_1: $\mu < -1$ which will make a Type I error with probability no greater than $\alpha = 0.01$.*

  (ii) *Do the same as in part (i), but supposing that σ is known.*

  (iii) *Give an intuitive explanation of why the critical values are different and say which test you think has greater power.*

Answer:

(i) Use the left-tailed t-test

$$\text{Reject } H_0 \text{ iff } \sqrt{20}(\bar{X}_{20} - (-1))/S_{20} < -t_{20-1,0.01} = \texttt{-qt(.99,19)} = -2.539483.$$

(ii) Use the left-tailed Z-test

$$\text{Reject } H_0 \text{ iff } \sqrt{20}(\bar{X}_{20} - (-1))/\sigma < -z_{0.01} = \texttt{-qnorm(.99)} = -2.326348.$$

(iii) The critical value when σ is unknown is greater in magnitude than when σ is known, so we can think of it as being more difficult to reject H_0 when σ is unknown, or that stronger evidence against H_0 is required. According to this reasoning, the power when σ is known should be greater.

• The power of the t-tests is not as simple to compute as that of the so-called Z-tests. This is because our estimate S_n^2 of the variance σ^2 is subject to random error. To compute the power of the t-tests, we must make use of a distribution called the *non-central t-distribution*.

• **Definition:** Let $Z \sim \text{Normal}(0,1)$ and $W \sim \chi_\nu^2$ be independent random variables and let ϕ be a constant. Then the distribution of the random variable defined by

$$T = \frac{Z + \phi}{\sqrt{W/\nu}}$$

is called the *non-central t-distribution* with ν degrees of freedom and *non-centrality parameter ϕ*.

• When $\phi = 0$ the non-central t-distributions are the same as the t-distributions.

• For $\xi \in (0,1)$, let $t_{\phi,n-1,\xi}$ be the value which satisfies $\xi = P(T > t_{\phi,n-1,\xi})$, when $T \sim t_{\phi,n-1}$.

• Let $F_{t_{\phi,n-1}}$ denote the cdf of the non-central t-distribution with $n-1$ degrees of freedom and non-centrality parameter ϕ.

• The pdfs and cdfs of the non-central t-distributions are very complicated. We will not worry about writing them down. We can get quantiles of the non-central t-distributions from R with the $\texttt{qt()}$ function. For example, $t_{1,19,0.05} = \texttt{qt(.95,19,1)} = 2.842502$.

• We will base our power calculations for the t-tests on the following result:

Result: Let $X_1, \ldots, X_n$ be a random sample from the $\text{Normal}(\mu, \sigma^2)$ distribution. Then

$$\frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \sim t_{\phi,n-1}, \text{ where } \phi = \frac{\mu - \mu_0}{\sigma/\sqrt{n}}.$$

Derivation: We can see the above by writing

$$\frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} = \left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} + \frac{\mu - \mu_0}{\sigma/\sqrt{n}} \right) \left[\frac{(n-1)S_n^2}{\sigma^2}/(n-1) \right]^{-1/2}$$

and recalling results from STAT 512.

13

- **Exercise:** *Get expressions for the power functions of the right-tailed, left-tailed, and two-sided t-tests.*

    Answers:

    1. The power function for the right-tailed t-test is

$$
\begin{aligned}
\gamma(\mu) &= P_\mu(\sqrt{n}(\bar{X}_n - \mu_0)/S_n > C_1) \\
&= P(T > C_1), \quad T \sim t_{\phi, n-1}, \quad \phi = \sqrt{n}(\mu - \mu_0)/\sigma \\
&= 1 - P(T < C_1) \\
&= 1 - F_{t_{\phi, n-1}}(C_1).
\end{aligned}
$$

    2. The power function for the left-tailed t-test is

$$
\begin{aligned}
\gamma(\mu) &= P_\mu(\sqrt{n}(\bar{X}_n - \mu_0)/\sigma < -C_1) \\
&= P(T < -C_1), \quad T \sim t_{\phi, n-1}, \quad \phi = \sqrt{n}(\mu - \mu_0)/\sigma \\
&= F_{t_{\phi, n-1}}(-C_1).
\end{aligned}
$$

    3. The power function for the two-sided t-test is

$$
\begin{aligned}
\gamma(\mu) &= P_\mu(|\sqrt{n}(\bar{X}_n - \mu_0)/\sigma| > C_2) \\
&= P(|T| > C_2), \quad T \sim t_{\phi, n-1}, \quad \phi = \sqrt{n}(\mu - \mu_0)/\sigma \\
&= 1 - P(-C_2 < T < C_2), \quad (t_{\phi, n-1} \text{ not symmetric for } \phi \neq 0) \\
&= 1 - [F_{t_{\phi, n-1}}(C_2) - F_{t_{\phi, n-1}}(-C_2)].
\end{aligned}
$$

- **Exercise:** *Let $X_1, \ldots, X_n$ be a random sample from the Normal(μ, σ^2) distribution, where μ and σ^2 are unknown, and suppose there are three researchers:*

    - *Efstathios to test H_0: $\mu \leq 5$ versus H_1: $\mu > 5$ with test Reject H_0 iff $\sqrt{n}(\bar{X}_n - 5)/S_n > t_{n-1, 0.10}$*
    - *Dimitris to test H_0: $\mu \geq 5$ versus H_1: $\mu < 5$ with test Reject H_0 iff $\sqrt{n}(\bar{X}_n - 5)/S_n < -t_{n-1, 0.10}$*
    - *Phoebe to test H_0: $\mu = 5$ versus H_1: $\mu \neq 5$ with test Reject H_0 iff $|\sqrt{n}(\bar{X}_n - 5)/2|/S_n > t_{n-1, 0.05}$*

    Each will collect a sample of size $n = 20$.

    (i) What is the size of each test?

    (ii) Compute the power of each researcher's test when $\mu = 4.5$ and $\sigma^2 = 4$.

(iii) *Plot the power curves of the three researchers' tests together when $\sigma^2 = 4$. Then add to the plot the power curves of the tests the researchers would use if the true value of σ^2 were known to them.*

(iv) *What can be said about the power curves?*

- **Answers:**

  (ii) In the following, let $T \sim t_{\phi,19}$, where $\phi = \sqrt{20}(4.5 - 5)/2$. Then:

  – The test of Efstathios has power

  $$1 - P(T > t_{19,0.10}) = \texttt{1 - pt(qt(.9,19),19,sqrt(20)*(4.5-5)/2)} = 0.008753802.$$

  – The test of Dimitris has power

  $$P(T < -t_{19,0.10}) = \texttt{pt(-qt(.9,19),19,sqrt(20)*(4.5-5)/2)} = 0.4254995.$$

  – The test of Phoebe has power

  $$1 - [P(T < t_{19,0.05}) - P(T < -t_{19,0.05})]$$
  $$= \texttt{1-(pt(qt(.95,19),19,sqrt(20)*(4.5-5)/2)}$$
  $$\texttt{-pt(-qt(.95,19),19,sqrt(20)*(4.5-5)/2))}$$
  $$= 0.2887311.$$

 (iii) The following R code makes the plot:

```
mu.seq <- seq(3,7,length=100)
mu.0 <- 5
n <- 20

power.E <- 1-pnorm(qnorm(.9)-sqrt(n)*(mu.seq - mu.0)/2)
power.D <- pnorm(-qnorm(.9)-sqrt(n)*(mu.seq - mu.0)/2)
power.H <- 1-(pnorm(qnorm(.95)-sqrt(n)*(mu.seq - mu.0)/2)
              - pnorm(-qnorm(.95)-sqrt(n)*(mu.seq - mu.0)/2))

power.Et <- 1 - pt(qt(.9,n-1),n-1,sqrt(n)*(mu.seq[j]-mu.0)/2)
power.Dt <- pt(-qt(.9,n-1),n-1,sqrt(n)*(mu.seq[j]-mu.0)/2)
power.Ht <- 1-(pt(qt(.95,n-1),n-1,sqrt(n)*(mu.seq-mu.0)/2)
              -pt(-qt(.95,n-1),n-1,sqrt(n)*(mu.seq-mu.0)/2))

plot(mu.seq, power.E,type="l",ylim=c(0,1),xlab="mu",ylab="power")
lines(mu.seq, power.D,lty=2)
lines(mu.seq, power.H,lty=4)
lines(mu.seq, power.Et,lty=1,col=rgb(0,0,.545))
lines(mu.seq, power.Dt,lty=2,col=rgb(0,0,.545))
lines(mu.seq, power.Ht,lty=4,col=rgb(0,0,.545))
abline(v=mu.0,lty=3) # vert line at null value
abline(h=0.10,lty=3) # horiz line at size
```
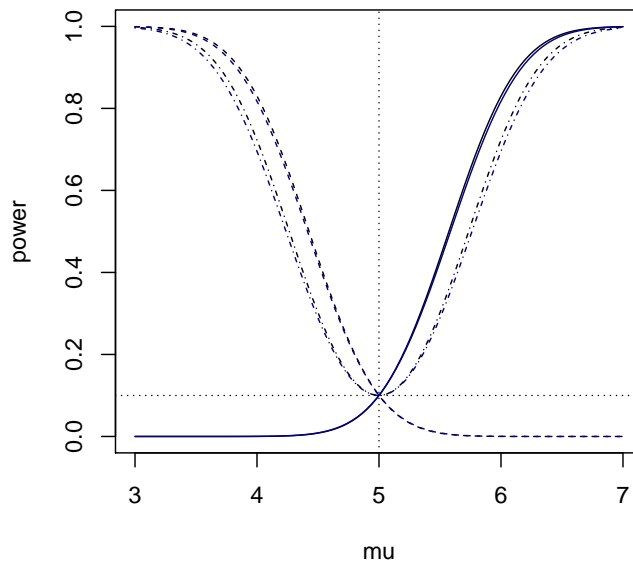

(iv) The power when $\mu \neq \mu_0$ of the t-tests (σ-unknown tests) is slightly lower than that of the Z-tests (σ-known tests). The difference in power would be larger for smaller sample sizes (try putting smaller n into the code!). This illustrates the price we pay for not knowing the

16

population standard deviation σ.

- **Formulas concerning tests about Normal mean (when variance unknown):** Suppose $X_1, \ldots, X_n$ is a random sample from the Normal(μ, σ^2) distribution, where μ and σ^2 are unknown. Then we have the following, where $F_{t_{\phi,n-1}}$ is the cdf of the $t_{\phi,n-1}$ distribution and $T_n = \sqrt{n}(\bar{X}_n - \mu_0)/S_n$:

H_0	H_1	Reject H_0 at α iff	Power function $\gamma(\mu)$
$\mu \leq \mu_0$	$\mu > \mu_0$	$T_n > t_{n-1,\alpha}$	$1 - F_{t_{\phi,n-1}}(t_{n-1,\alpha})$
$\mu \geq \mu_0$	$\mu < \mu_0$	$T_n < -t_{n-1,\alpha}$	$F_{t_{\phi,n-1}}(-t_{n-1,\alpha})$
$\mu = \mu_0$	$\mu \neq \mu_0$	$|T_n| > t_{n-1,\alpha/2}$	$1 - [F_{t_{\phi,n-1}}(t_{n-1,\alpha/2}) - F_{t_{\phi,n-1}}(-t_{n-1,\alpha/2})],$

where the value of the noncentrality parameter is given by $\phi = \sqrt{n}(\mu - \mu_0)/\sigma$.

Inference about the variance of a Normal population

- To make inferences about the variance σ^2 of a Normal population, we will make use of this pivot quantity result: If $X_1, \ldots, X_n$ is a random sample from the Normal(μ, σ^2) distribution, then

$$\frac{(n-1)S_n}{\sigma^2} \sim \chi^2_{n-1}.$$

- Let $F_{\chi^2_{n-1}}$ be the cdf of the χ^2-distribution with $n-1$ degrees of freedom.

- **Exercise:** *Suppose $X_1, \ldots, X_{15}$ are the heights of a random sample of 2-yr-old trees on a tree farm, and assume that the heights follow a Normal distribution. Researchers plan to test H_0: $\sigma \geq 3$ versus H_1: $\sigma < 3$ with the test*

$$\text{Reject } H_0 \text{ iff } S_{15} < 2.$$

- (i) *Suppose the true value of σ is $\sigma = 2.5$. With what probability will the researchers reject H_0?*
- (ii) *Find an expression for the power $\gamma(\sigma)$ of the test for any value of σ.*
- (iii) *Over all possible true values of σ find the maximum probability that the researchers will commit a Type I error.*
- (iv) *Make a plot of the power function $\gamma(\sigma)$ of the test over $\sigma \in (1, 4)$.*

Answers:

(i)   If $\sigma = 2.5$ the researchers will reject H_0 with probability

$$
\begin{aligned}
P_{\sigma=2.5}(S_{15} < 2) &= P_{\sigma=2.5}(S_{15}^2 < 4) \\
&= P_{\sigma=2.5}((15-1)S_{15}^2/(2.5)^2 < 4(15-1)/(2.5)^2) \\
&= P(W < 4(15-1)/(2.5)^2), \quad W \sim \chi_{14}^2 \\
&= F_{\chi_{14}^2}(4(15-1)/(2.5)^2) \\
&= \texttt{pchisq(4*(15-1)/(2.5)**2,14)} \\
&= 0.1663956.
\end{aligned}
$$

(ii) The power is given by

$$
\begin{aligned}
\gamma(\sigma) &= P_\sigma(S_{15} < 2) \\
&= P_\sigma((15-1)S_{15}^2/\sigma^2 < 4(15-1)/\sigma^2) \\
&= P(W < 4(15-1)/\sigma^2), \quad W \sim \chi_{14}^2 \\
&= F_{\chi_{14}^2}(4(15-1)/\sigma^2)
\end{aligned}
$$

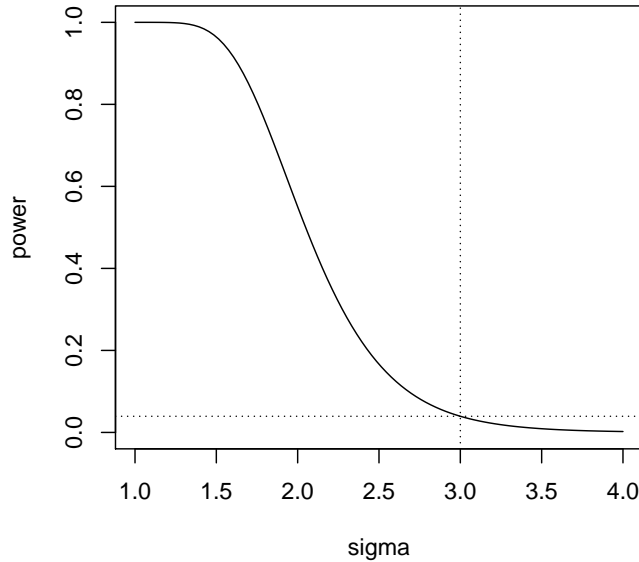(iii) This is the size of the test, which is given by

$$
\begin{aligned}
\sup_{\sigma \geq 3} \gamma(\sigma) &= \gamma(3) \\
&= F_{\chi_{14}^2}(4(15-1)/9) \\
&= \texttt{pchisq(4*(15-1)/9,14)} \\
&= 0.03942441.
\end{aligned}
$$

(iv) The following R code makes the plot:

```
sigma.seq <- seq(1,4,length=100)
sigma.0 <- 3

power <- pchisq(4*(15-1)/sigma.seq^2,15-1)

plot(sigma.seq, power,type="l",ylim=c(0,1),xlab="sigma",ylab="power")
abline(v=sigma.0,lty=3)     # vert line at null value
abline(h=0.03942441,lty=3)  # horiz line at size
```

18

- Concerning the variance of a Normal population, we will consider, for some constants C_{1r}, C_{1l}, C_{2r}, and C_{2l}, tests like the following:

  1. *Right-tailed test:* Test H_0: $\sigma^2 \leq \sigma_0^2$ versus H_1: $\sigma^2 > \sigma_0^2$ with the test

  $$\text{Reject } H_0 \text{ iff } \frac{(n-1)S_n^2}{\sigma_0^2} > C_{1r}.$$

  2. *Left-tailed test:* Test H_0: $\sigma^2 \geq \sigma_0^2$ versus H_1: $\sigma^2 < \sigma_0^2$ with the test

  $$\text{Reject } H_0 \text{ iff } \frac{(n-1)S_n^2}{\sigma_0^2} < C_{1l}.$$

  3. *Two-sided test:* Test H_0: $\sigma^2 = \sigma_0^2$ versus H_1: $\sigma^2 \neq \sigma_0^2$ with the test

  $$\text{Reject } H_0 \text{ iff } \frac{(n-1)S_n^2}{\sigma_0^2} < C_{2l} \text{ or } \frac{(n-1)S_n^2}{\sigma_0^2} > C_{2r}.$$

- **Exercise:** *Get expressions for the power functions of the above tests and calibrate the rejection regions, that is, find C_{1l}, C_{1r}, C_{2l}, and C_{2r}, such that for any $\alpha \in (0,1)$ each test has size α.*

19

Answers:

1. The right-tailed test has power given by

$$
\begin{aligned}
\gamma(\sigma^2) &= P_{\sigma^2}((n-1)S_n^2/\sigma_0^2 > C_{1r}) \\
&= P_{\sigma^2}((\sigma^2/\sigma_0^2)(n-1)S_n^2/\sigma^2 > C_{1r}) \\
&= P_{\sigma^2}((n-1)S_n^2/\sigma^2 > C_{1r}(\sigma_0^2/\sigma^2)) \\
&= P(W > C_{1r}(\sigma_0^2/\sigma^2)), \quad W \sim \chi_{n-1}^2 \\
&= 1 - F_{\chi_{n-1}^2}(C_{1r}(\sigma_0^2/\sigma^2)).
\end{aligned}
$$

The size is given by

$$
\sup_{\sigma^2 \leq \sigma_0^2} \gamma(\sigma^2) = \gamma(\sigma_0^2) = 1 - F_{\chi_{n-1}^2}(C_{1r}).
$$

The test will have size equal to α if $C_{1r} = \chi_{n-1,\alpha}^2$.

2. The left-tailed test has power given by

$$
\begin{aligned}
\gamma(\sigma^2) &= P_{\sigma^2}((n-1)S_n^2/\sigma_0^2 < C_{1l}) \\
&= P_{\sigma^2}((\sigma^2/\sigma_0^2)(n-1)S_n^2/\sigma^2 < C_{1l}) \\
&= P_{\sigma^2}((n-1)S_n^2/\sigma^2 < C_{1l}(\sigma_0^2/\sigma^2)) \\
&= P(W < C_{1l}(\sigma_0^2/\sigma^2)), \quad W \sim \chi_{n-1}^2 \\
&= F_{\chi_{n-1}^2}(C_{1l}(\sigma_0^2/\sigma^2)).
\end{aligned}
$$

The size is given by

$$
\sup_{\sigma^2 \geq \sigma_0^2} \gamma(\sigma^2) = \gamma(\sigma_0^2) = F_{\chi_{n-1}^2}(C_{1l}).
$$

The test will have size equal to α if $C_{1l} = \chi_{n-1,1-\alpha}^2$.

3. The two-sided test has power given by

$$
\begin{aligned}
\gamma(\sigma^2) &= P_{\sigma^2}((n-1)S_n^2/\sigma_0^2 < C_{2l}) + P_{\sigma^2}((n-1)S_n^2/\sigma_0^2 > C_{2r}) \\
&= P_{\sigma^2}((\sigma^2/\sigma_0^2)(n-1)S_n^2/\sigma^2 < C_{2l}) + P_{\sigma^2}((\sigma^2/\sigma_0^2)(n-1)S_n^2/\sigma^2 > C_{2r}) \\
&= P_{\sigma^2}((n-1)S_n^2/\sigma^2 < C_{2l}(\sigma_0^2/\sigma^2)) + P_{\sigma^2}((n-1)S_n^2/\sigma^2 > C_{2r}(\sigma_0^2/\sigma^2)) \\
&= P(W < C_{2l}(\sigma_0^2/\sigma^2)) + P(W > C_{2r}(\sigma_0^2/\sigma^2)), \quad W \sim \chi_{n-1}^2 \\
&= F_{\chi_{n-1}^2}(C_{2l}(\sigma_0^2/\sigma^2)) + 1 - F_{\chi_{n-1}^2}(C_{2r}(\sigma_0^2/\sigma^2)).
\end{aligned}
$$

The size is given by

$$
\sup_{\sigma^2 \in \{\sigma_0^2\}} \gamma(\sigma^2) = \gamma(\sigma_0^2) = F_{\chi_{n-1}^2}(C_{2l}) + 1 - F_{\chi_{n-1}^2}(C_{2r}).
$$

The test will have size equal to α if $C_{2l} = \chi_{n-1,1-\alpha/2}^2$ and $C_{2r} = \chi_{n-1,\alpha/2}^2$.
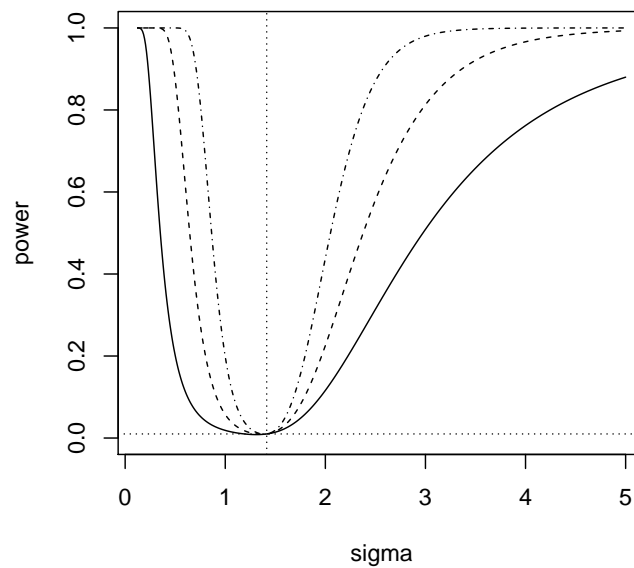
- **Exercise:** *Suppose $X_1, \ldots, X_n$ is a random sample from the Normal(μ, σ^2) distribution where μ and σ are unknown. Researchers are interested in testing H_0: $\sigma^2 = 2$ versus H_1: $\sigma \neq 2$. Make a plot showing the power in terms of σ of the two-sided test with the size set to $\alpha = 0.01$ for the sample sizes $n = 5, 10, 20$.*

Answer: The following R code makes the plot:

```
sigma.seq <- seq(1/8,5,length=500)
sigma.0 <- sqrt(2)

power.n5 <- pchisq(qchisq(.005,5-1)*sigma.0^2/sigma.seq^2,5-1)
              + 1 - pchisq(qchisq(.995,5-1)*sigma.0^2/sigma.seq^2,5-1)
power.n10 <- pchisq(qchisq(.005,10-1)*sigma.0^2/sigma.seq^2,10-1)
              + 1 - pchisq(qchisq(.995,10-1)*sigma.0^2/sigma.seq^2,10-1)
power.n20 <-pchisq(qchisq(.005,20-1)*sigma.0^2/sigma.seq^2,20-1)
              + 1 - pchisq(qchisq(.995,20-1)*sigma.0^2/sigma.seq^2,20-1)

plot(sigma.seq, power.n5,type="l",ylim=c(0,1),xlab="sigma",ylab="power")
lines(sigma.seq, power.n10,lty=2)
lines(sigma.seq, power.n20,lty=4)
abline(v=sigma.0,lty=3) # vert line at null value
abline(h=0.01,lty=3)        # horiz line at size
```

- **Formulas concerning tests about Normal variance:** Suppose $X_1, \ldots, X_n$ is a random sample from the Normal(μ, σ^2) distribution, where μ and σ^2 are unknown. Then we have the following, where $F_{\chi^2_{n-1}}$ is the cdf of the χ^2_{n-1}-distribution and $W_n = (n-1)S_n^2/\sigma_0^2$:

H_0	H_1	Reject H_0 at α iff	Power function $\gamma(\sigma^2)$
$\sigma^2 \leq \sigma_0^2$	$\sigma^2 > \sigma_0^2$	$W_n > \chi^2_{n-1,\alpha}$	$1 - F_{\chi^2_{n-1}}(\chi^2_{n-1,\alpha}(\sigma_0^2/\sigma^2))$
$\sigma^2 \geq \sigma_0^2$	$\sigma^2 < \sigma_0^2$	$W_n < \chi^2_{n-1,1-\alpha}$	$F_{\chi^2_{n-1}}(\chi^2_{n-1,1-\alpha}(\sigma_0^2/\sigma^2))$
$\sigma^2 = \sigma_0^2$	$\sigma^2 \neq \sigma_0^2$	$W_n < \chi^2_{n-1,1-\alpha/2}$ or $W_n > \chi^2_{n-1,\alpha/2}$	$F_{\chi^2_{n-1}}(\chi^2_{n-1,\alpha/2}(\sigma_0^2/\sigma^2))$ $+ 1 - F_{\chi^2_{n-1}}(\chi^2_{n-1,1-\alpha/2}(\sigma_0^2/\sigma^2))$

Connection between two-sided tests and confidence intervals

- If $X_1, \ldots, X_n$ is a random sample from the Normal(μ, σ^2) distribution, where μ is unknown *but σ is known*, then a size-α test of H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$ is

$$\text{Reject } H_0 \text{ iff } |\sqrt{n}(\bar{X}_n - \mu_0)/\sigma| > z_{\alpha/2}.$$

Recall that a $(1-\alpha)100\%$ confidence interval for μ is $\bar{X}_n \pm z_{\alpha/2}\sigma/\sqrt{n}$. We find

$$|\sqrt{n}(\bar{X}_n - \mu_0)/\sigma| < z_{\alpha/2} \text{ iff } \mu_0 \in (\bar{X}_n - z_{\alpha/2}\sigma/\sqrt{n}, \bar{X}_n + z_{\alpha/2}\sigma/\sqrt{n}),$$

that is, we fail to reject H_0 iff the null value lies in the confidence interval, so an equivalent test is

$$\text{Reject } H_0 \text{ iff } \mu_0 \notin (\bar{X}_n - z_{\alpha/2}\sigma/\sqrt{n}, \bar{X}_n + z_{\alpha/2}\sigma/\sqrt{n}).$$

- If $X_1, \ldots, X_n$ is a random sample from the Normal(μ, σ^2) distribution, where μ is unknown *and σ is unknown*, then a size-α test of H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$ is

$$\text{Reject } H_0 \text{ iff } |\sqrt{n}(\bar{X}_n - \mu_0)/S_n| > t_{n-1,\alpha/2}.$$

Recall that a $(1-\alpha)100\%$ confidence interval for μ is $\bar{X}_n \pm t_{n-1,\alpha/2}S_n/\sqrt{n}$. We find

$$|\sqrt{n}(\bar{X}_n - \mu_0)/S_n| < t_{n-1,\alpha/2} \text{ iff } \mu_0 \in (\bar{X}_n - t_{n-1,\alpha/2}S_n/\sqrt{n}, \bar{X}_n + t_{n-1,\alpha/2}S_n/\sqrt{n}),$$

that is, we fail to reject H_0 iff the null value lies in the confidence interval, so an equivalent test is

$$\text{Reject } H_0 \text{ iff } \mu_0 \notin (\bar{X}_n - t_{n-1,\alpha/2}S_n/\sqrt{n}, \bar{X}_n + t_{n-1,\alpha/2}S_n/\sqrt{n}).$$

- If $X_1, \ldots, X_n$ is a random sample from the Normal(μ, σ^2) distribution, where μ and σ are unknown, then a size-α test of H_0: $\sigma^2 = \sigma_0^2$ versus H_1: $\sigma^2 \neq \sigma_0^2$ is

$$\text{Reject } H_0 \text{ iff } \frac{(n-1)S_n^2}{\sigma_0^2} < \chi^2_{n-1,1-\alpha/2} \text{ or } \frac{(n-1)S_n^2}{\sigma_0^2} > \chi^2_{n-1,\alpha/2}$$

Recall that a $(1-\alpha)100\%$ confidence interval for σ^2 is $((n-1)S_n^2/\chi^2_{n-1,\alpha/2}, (n-1)S_n^2/\chi^2_{n-1,1-\alpha/2})$. We find

$$\chi^2_{n-1,1-\alpha/2} < \frac{(n-1)S_n^2}{\sigma_0^2} < \chi^2_{n-1,\alpha/2} \text{ iff } \sigma_0^2 \in ((n-1)S_n^2/\chi^2_{n-1,\alpha/2}, (n-1)S_n^2/\chi^2_{n-1,1-\alpha/2}),$$

that is, we fail to reject H_0 iff the null value lies in the confidence interval, so an equivalent test is

$$\text{Reject } H_0 \text{ iff } \sigma_0^2 \notin ((n-1)S_n^2/\chi^2_{n-1,\alpha/2}, (n-1)S_n^2/\chi^2_{n-1,1-\alpha/2}).$$