STAT 513 fa 2020 Lec 02 slides

Inference about the mean and variance of a Normal population

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

- Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$.
- We wish to test hypotheses about μ and σ^2 based on $X_1, \dots, X_n$.
- We will consider
  1. tests about μ when σ^2 is known
  2. tests about μ when σ^2 is unknown
  3. tests about σ^2 when μ is unknown

Pivot quantity results from STAT 512

If $X_1, \ldots, X_n \stackrel{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, then

1. $\sqrt{n}(\bar{X}_n - \mu)/\sigma \sim \text{Normal}(0, 1)$
2. $\sqrt{n}(\bar{X}_n - \mu)/S_n \sim t_{n-1}$
3. $(n-1)S_n^2/\sigma^2 \sim \chi^2_{n-1}$

In the above

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i \quad \text{and} \quad S_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$$

Tests about μ when σ^2 is known:

1. *Right-tailed test*: Test H_0: $\mu \leq \mu_0$ versus H_1: $\mu > \mu_0$ with the test

$$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} > C_1.$$

2. *Left-tailed test*: Test H_0: $\mu \geq \mu_0$ versus H_1: $\mu < \mu_0$ with the test

$$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} < -C_1.$$

3. *Two-sided test*: Test H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$ with the test

$$\text{Reject } H_0 \text{ iff } \left|\frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}}\right| > C_2.$$

We often refer to values playing the role of C_1, $-C_1$, and C_2 as *critical values*.

Exercise: For the right-tailed, left-tailed, and two-sided test on the previous slide:

1. Get an expression for the power function.
2. For any $\alpha \in (0, 1)$, give the value of C_1 or C_2 such that the test has size α.

Tests about μ when σ^2 is known: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, σ^2 known.

For some null value μ_0, define the test statistic

$$Z_n = \sqrt{n}(\bar{X}_n - \mu_0)/\sigma.$$

Then we have the following:

H_0	H_1	Reject H_0 at α iff	Power function $\gamma(\mu)$
$\mu \leq \mu_0$	$\mu > \mu_0$	$Z_n > z_\alpha$	$1 - \Phi(z_\alpha - \sqrt{n}(\mu - \mu_0)/\sigma)$.
$\mu \geq \mu_0$	$\mu < \mu_0$	$Z_n < -z_\alpha$	$\Phi(-z_\alpha - \sqrt{n}(\mu - \mu_0)/\sigma)$
$\mu = \mu_0$	$\mu \neq \mu_0$	$\lvert Z_n \rvert > z_{\alpha/2}$	$1 - [\Phi(z_{\alpha/2} - \sqrt{n}(\mu - \mu_0)/\sigma)$ $-\Phi(-z_{\alpha/2} - \sqrt{n}(\mu - \mu_0)/\sigma)]$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, 4)$. Suppose

- Efstathios will test $H_0\colon \mu \leq 5$ versus $H_1\colon \mu > 5$ with

$$\text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - 5)/2 > z_{0.10}$$

- Dimitris will test $H_0\colon \mu \geq 5$ versus $H_1\colon \mu < 5$ with

$$\text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - 5)/2 < -z_{0.10}$$

- Phoebe will test $H_0\colon \mu = 5$ versus $H_1\colon \mu \neq 5$ with

$$\text{Reject } H_0 \text{ iff } |\sqrt{n}(\bar{X}_n - 5)/2| > z_{0.05}$$

Each will collect a sample of size $n = 20$.

1. If in truth $\mu = 4.5$, who may commit a Type I error?
2. If in truth $\mu = 5.5$, who may commit a Type I error?
3. If in truth $\mu = 4.5$, who may commit a Type II error?
4. If in truth $\mu = 5.5$, who may commit a Type II error?

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, 4)$. Suppose

- Efstathios will test $H_0\colon \mu \leq 5$ versus $H_1\colon \mu > 5$ with

$$\text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - 5)/2 > z_{0.10}$$

- Dimitris will test $H_0\colon \mu \geq 5$ versus $H_1\colon \mu < 5$ with

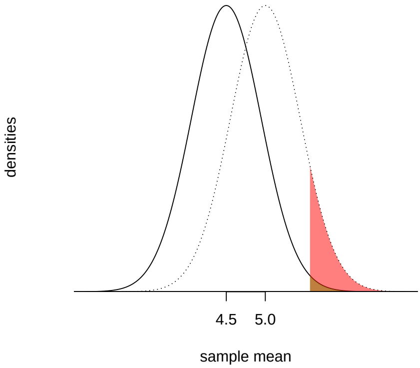
$$\text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - 5)/2 < -z_{0.10}$$

- Phoebe will test $H_0\colon \mu = 5$ versus $H_1\colon \mu \neq 5$ with

$$\text{Reject } H_0 \text{ iff } |\sqrt{n}(\bar{X}_n - 5)/2| > z_{0.05}$$

Each will collect a sample of size $n = 20$.

5. If in truth $\mu = 4.5$, who is most likely to reject H_0?
6. If in truth $\mu = 4.5$, who is least likely to reject H_0?
7. Compute the power of each researcher's test when $\mu = 4.5$.
8. Plot the power curves of the three researchers' tests together.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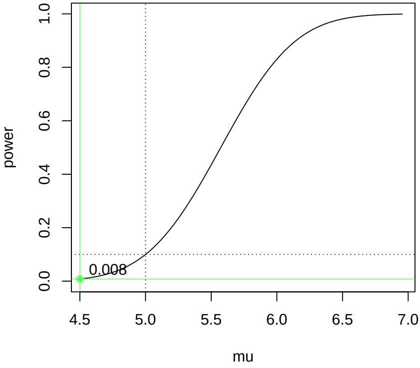
5.0
sample mean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0.6
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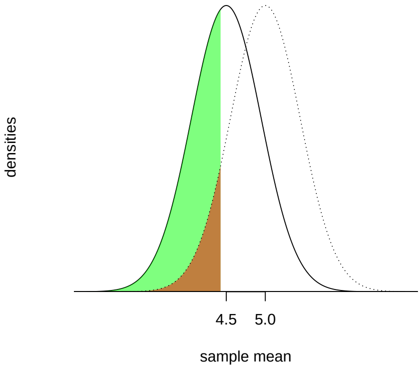
5.0
5.5
6.0
6.5
7.0
mu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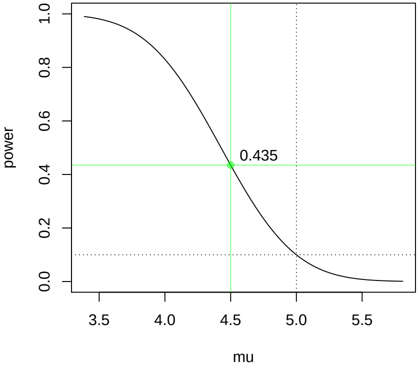
5.0
sample mean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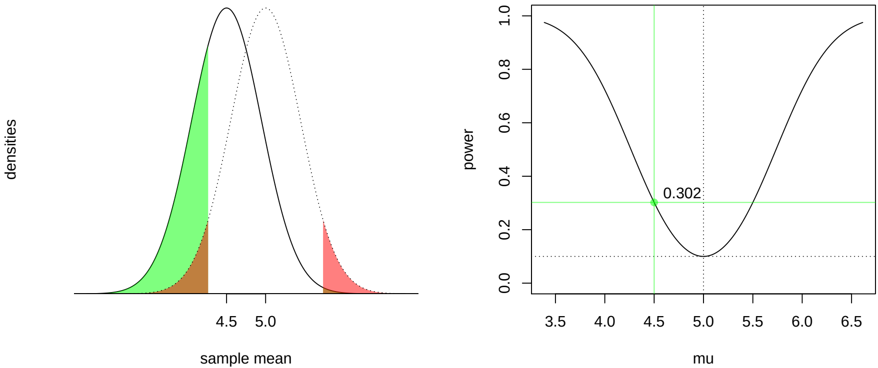
0.6
0.4
0.2
0.0
0.435
3.5
4.0
4.5
5.0
5.5
mu


densities
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5.0
sample mean
power
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0.2
0.4
0.6
0.8
1.0
0.302
3.5
4.0
4.5
5.0
5.5
6.0
6.5
mu

Tests about μ when σ^2 is unknown: Use S_n instead of σ.

1. *Right-tailed test*: Test H_0: $\mu \leq \mu_0$ versus H_1: $\mu > \mu_0$ with the test

$$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} > C_1.$$

2. *Left-tailed test*: Test H_0: $\mu \geq \mu_0$ versus H_1: $\mu < \mu_0$ with the test

$$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} < -C_1.$$

3. *Two-sided test*: Test H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$ with the test

$$\text{Reject } H_0 \text{ iff } \left| \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \right| > C_2.$$

Exercise: For any $\alpha \in (0, 1)$, find C_1 and C_2 such that these have size α.

Exercise: Let $X_1, \dots, X_{20} \stackrel{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$.

1. Give a test of H_0: $\mu \geq -1$ versus H_1: $\mu < -1$ which will make a Type I error with probability no greater than $\alpha = 0.01$.
2. Do the same as in part (i), but supposing that σ is known.
3. Discuss why the critical values are different and which test has greater power.

Power of the t-tests is more complicated to compute than that of the Z-tests...

Non-central t-distribution

Let $Z \sim \text{Normal}(0,1)$ and $W \sim \chi^2_\nu$ be independent rvs and let ϕ be a constant.

Then the distribution of the rv

$$T = \frac{Z + \phi}{\sqrt{W/\nu}}$$

is called the *non-central t-distribution* with df ν and *non-centrality parameter* ϕ.

Denote the distribution by $t_{\phi,\nu}$. When $\phi = 0$, the non-central t-distributions are the same as the t-distributions.

For $\xi \in (0,1)$, let $t_{\phi,\nu,\xi}$ be value s.t. $\xi = P(T > t_{\phi,\nu,\xi})$ when $T \sim t_{\phi,\nu}$.

Let $F_{t_{\phi,\nu}}$ denote the cdf of the non-central t-dist. with df ν and ncp ϕ.

Non-central t-distribution pivot quantity result

If $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, then

$$\sqrt{n}(\bar{X}_n - \mu_0)/S_n \sim t_{\phi, n-1},$$

with $\phi = \sqrt{n}(\mu - \mu_0)/\sigma$.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$ and consider

$$H_0\colon \mu \leq \mu_0 \text{ versus } H_1\colon \mu > \mu_0.$$

Get an expression for the power function of the test

$$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} > t_{n-1,\alpha}.$$

Tests about μ when σ^2 unknown: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, σ^2 unknown.

For some null value μ_0, define the test statistic

$$T_n = \sqrt{n}(\bar{X}_n - \mu_0)/S_n.$$

Then we have the following:

H_0	H_1	Reject H_0 at α iff	Power function $\gamma(\mu)$
$\mu \leq \mu_0$	$\mu > \mu_0$	$T_n > t_{n-1,\alpha}$	$1 - F_{t_{\phi,n-1}}(t_{n-1,\alpha})$
$\mu \geq \mu_0$	$\mu < \mu_0$	$T_n < -t_{n-1,\alpha}$	$F_{t_{\phi,n-1}}(-t_{n-1,\alpha})$
$\mu = \mu_0$	$\mu \neq \mu_0$	$\lvert T_n \rvert > t_{n-1,\alpha/2}$	$1 - [F_{t_{\phi,n-1}}(t_{n-1,\alpha/2}) - F_{t_{\phi,n-1}}(-t_{n-1,\alpha/2})]$,

In the above, $\phi = \sqrt{n}(\mu - \mu_0)/\sigma$.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$. Suppose

- Efstathios will test H_0: $\mu \leq 5$ vs H_1: $\mu > 5$ with

$$\text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - 5)/S_n > t_{n-1,0.10}$$

- Dimitris will test H_0: $\mu \geq 5$ vs H_1: $\mu < 5$ with

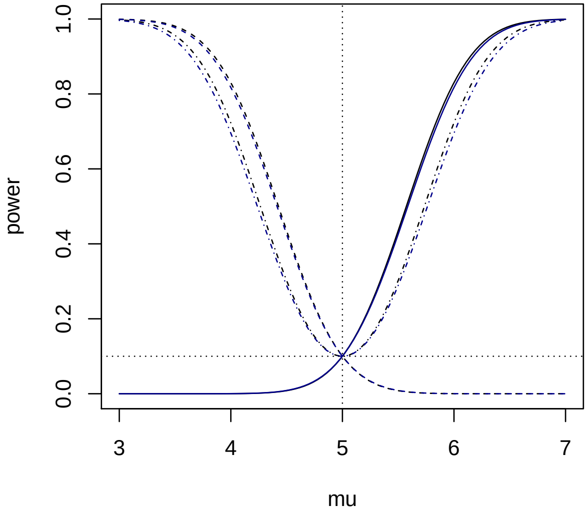
$$\text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - 5)/S_n < -t_{n-1,0.10}$$

- Phoebe will test H_0: $\mu = 5$ vs H_1: $\mu \neq 5$ with

$$\text{Reject } H_0 \text{ iff } |\sqrt{n}(\bar{X}_n - 5)/2|/S_n > t_{n-1,0.05}$$

Each will collect a sample of size $n = 20$.

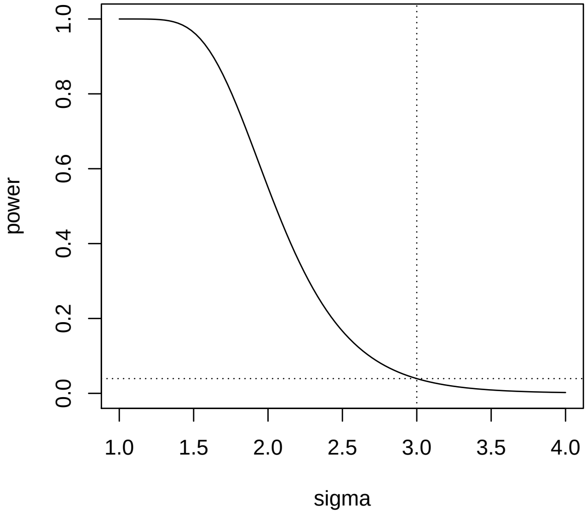
1. What is the size of each test?
2. Compute the power of each researcher's test when $\mu = 4.5$ and $\sigma^2 = 4$.
3. Plot the power curves of the three researchers' tests together when $\sigma^2 = 4$. Then overlay the σ^2-known versions.
4. What can be said about the power curves?


power
mu
0.0
0.2
0.4
0.6
0.8
1.0
3
4
5
6
7

Exercise: Let $X_1, \dots, X_{15}$ be heights of randomly selected 2-yr-old trees; assume they are Normal. Researchers will test H_0: $\sigma \geq 3$ versus H_1: $\sigma < 3$ with

$$\text{Reject } H_0 \text{ iff } S_{15} < 2.$$

1. If in truth $\sigma = 2.5$, with what probability will the test reject H_0?
2. Find an expression for the power $\gamma(\sigma)$ of the test for any value of σ.
3. Over all $\sigma > 0$ find the maximum probability of a Type I error.
4. Make a plot of the power function $\gamma(\sigma)$ of the test over $\sigma \in (1, 4)$.


power
0.0
0.2
0.4
0.6
0.8
1.0
1.0
1.5
2.0
2.5
3.0
3.5
4.0
sigma

Tests about σ^2:

- *Right-tailed test*: Test H_0: $\sigma^2 \leq \sigma_0^2$ versus H_1: $\sigma^2 > \sigma_0^2$ with the test

$$\text{Reject } H_0 \text{ iff } \frac{(n-1)S_n^2}{\sigma_0^2} > C_{1r}.$$

- *Left-tailed test*: Test H_0: $\sigma^2 \geq \sigma_0^2$ versus H_1: $\sigma^2 < \sigma_0^2$ with the test

$$\text{Reject } H_0 \text{ iff } \frac{(n-1)S_n^2}{\sigma_0^2} < C_{1l}.$$

- *Two-sided test*: Test H_0: $\sigma^2 = \sigma_0^2$ versus H_1: $\sigma^2 \neq \sigma_0^2$ with the test

$$\text{Reject } H_0 \text{ iff } \frac{(n-1)S_n^2}{\sigma_0^2} < C_{2l} \text{ or } \frac{(n-1)S_n^2}{\sigma_0^2} > C_{2r}.$$

Exercise: For any $\alpha \in (0, 1)$:

1. Find values C_{1r}, C_{1l}, and C_{2l} and C_{2r} such that these tests have size α.
2. Find expressions for the power functions of the size-α tests.

Tests about σ^2: Let $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$.

For some null value σ_0^2, define the test statistic

$$W_n = (n-1)S_n^2/\sigma_0^2.$$

Then we have the following:

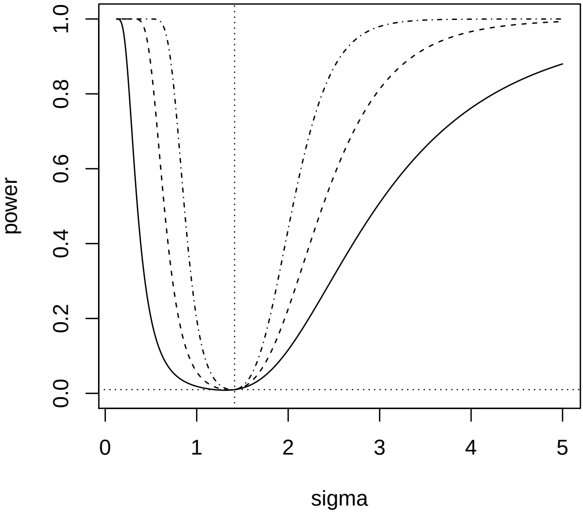
H_0	H_1	Reject H_0 at α iff	Power function $\gamma(\sigma^2)$
$\sigma^2 \leq \sigma_0^2$	$\sigma^2 > \sigma_0^2$	$W_n > \chi^2_{n-1,\alpha}$	$1 - F_{\chi^2_{n-1}}(\chi^2_{n-1,\alpha}(\sigma_0^2/\sigma^2))$
$\sigma^2 \geq \sigma_0^2$	$\sigma^2 < \sigma_0^2$	$W_n < \chi^2_{n-1,1-\alpha}$	$F_{\chi^2_{n-1}}(\chi^2_{n-1,1-\alpha}(\sigma_0^2/\sigma^2))$
$\sigma^2 = \sigma_0^2$	$\sigma^2 \neq \sigma_0^2$	$W_n < \chi^2_{n-1,1-\alpha/2}$ or $W_n > \chi^2_{n-1,\alpha/2}$	$F_{\chi^2_{n-1}}(\chi^2_{n-1,\alpha/2}(\sigma_0^2/\sigma^2))$ $+ 1 - F_{\chi^2_{n-1}}(\chi^2_{n-1,1-\alpha/2}(\sigma_0^2/\sigma^2))$

In the above $F_{\chi^2_{n-1}}$ is the cdf of the χ^2_{n-1}-distribution.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$ and

$$H_0\colon \sigma^2 = 2 \text{ versus } H_1\colon \sigma^2 \neq 2.$$

Plot the power as a function of σ of the two-sided test with the size $\alpha = 0.01$ under the sample sizes $n = 5, 10, 20$.


power
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5
sigma

Equivalence between CIs and two-sided tests for the mean:

If $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, a size-$\alpha$ test of H_0: $\mu = \mu_0$ vs H_1: $\mu \neq \mu_0$ is

$$\text{Reject } H_0 \text{ iff } \left| \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \right| > t_{n-1,\alpha/2}.$$

Recall that a $(1-\alpha) \times 100\%$ confidence interval for μ is given by

$$\bar{X}_n \pm t_{n-1,\alpha/2} S_n/\sqrt{n}.$$

Equivalent test is to check whether μ_0 contained in CI:

$$\text{Reject } H_0 \text{ iff } \mu_0 \notin (\bar{X}_n - t_{n-1,\alpha/2} S_n/\sqrt{n}, \bar{X}_n + t_{n-1,\alpha/2} S_n/\sqrt{n}).$$

Equivalence between CIs and two-sided tests for the variance:

If $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, a size-$\alpha$ test of H_0: $\sigma^2 = \sigma_0^2$ vs H_1: $\sigma^2 \neq \sigma_0^2$ is

$$\text{Reject } H_0 \text{ iff } \frac{(n-1)S_n^2}{\sigma_0^2} < \chi^2_{n-1,1-\alpha/2} \text{ or } \frac{(n-1)S_n^2}{\sigma_0^2} > \chi^2_{n-1,\alpha/2}.$$

Recall that a $(1-\alpha) \times 100\%$ confidence interval for μ is given by

$$\left(\frac{(n-1)S_n^2}{\chi^2_{n-1,\alpha/2}}, \frac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha/2}} \right)$$

Equivalent test is to check whether σ_0^2 contained in CI:

$$\text{Reject } H_0 \text{ iff } \sigma_0^2 \notin \left(\frac{(n-1)S_n^2}{\chi^2_{n-1,\alpha/2}}, \frac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha/2}} \right).$$