Inference about the mean and variance of a Normal population

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.
Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$.

We wish to test hypotheses about $\mu$ and $\sigma^2$ based on $X_1, \ldots, X_n$.

We will consider

1. tests about $\mu$ when $\sigma^2$ is known
2. tests about $\mu$ when $\sigma^2$ is unknown
3. tests about $\sigma^2$ when $\mu$ is unknown
Pivot quantity results from STAT 512

If $X_1, \ldots, X_n \sim \text{Normal}(\mu, \sigma^2)$, then

1. $\sqrt{n}(\bar{X}_n - \mu)/\sigma \sim \text{Normal}(0, 1)$

2. $\sqrt{n}(\bar{X}_n - \mu)/S_n \sim t_{n-1}$

3. $(n - 1)S_n^2/\sigma^2 \sim \chi^2_{n-1}$

In the above

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and} \quad S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$
Tests about $\mu$ when $\sigma^2$ is known:

1. **Right-tailed test**: Test $H_0: \mu \leq \mu_0$ versus $H_1: \mu > \mu_0$ with the test

   \[
   \text{Reject } H_0 \text{ iff } \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} > C_1.
   \]

2. **Left-tailed test**: Test $H_0: \mu \geq \mu_0$ versus $H_1: \mu < \mu_0$ with the test

   \[
   \text{Reject } H_0 \text{ iff } \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} < -C_1.
   \]

3. **Two-sided test**: Test $H_0: \mu = \mu_0$ versus $H_1: \mu \neq \mu_0$ with the test

   \[
   \text{Reject } H_0 \text{ iff } \left| \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} \right| > C_2.
   \]

We often refer to values playing the role of $C_1$, $-C_1$, and $C_2$ as **critical values**.
Exercise: For the right-tailed, left-tailed, and two-sided test on the previous slide:

1. Get an expression for the power function.
2. For any $\alpha \in (0, 1)$, give the value of $C_1$ or $C_2$ such that the test has size $\alpha$. 
Tests about \( \mu \) when \( \sigma^2 \) is known: Let \( X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2) \), \( \sigma^2 \) known.

For some null value \( \mu_0 \), define the test statistic

\[
Z_n = \sqrt{n}(\bar{X}_n - \mu_0)/\sigma.
\]

Then we have the following:

<table>
<thead>
<tr>
<th>( H_0 )</th>
<th>( H_1 )</th>
<th>Reject ( H_0 ) at ( \alpha ) iff</th>
<th>Power function ( \gamma(\mu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu \leq \mu_0 )</td>
<td>( \mu &gt; \mu_0 )</td>
<td>( Z_n &gt; z_\alpha )</td>
<td>( 1 - \Phi(z_\alpha - \sqrt{n}(\mu - \mu_0)/\sigma). )</td>
</tr>
<tr>
<td>( \mu \geq \mu_0 )</td>
<td>( \mu &lt; \mu_0 )</td>
<td>( Z_n &lt; -z_\alpha )</td>
<td>( \Phi(-z_\alpha - \sqrt{n}(\mu - \mu_0)/\sigma) )</td>
</tr>
<tr>
<td>( \mu = \mu_0 )</td>
<td>( \mu \neq \mu_0 )</td>
<td>(</td>
<td>Z_n</td>
</tr>
</tbody>
</table>
Exercise: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, 4)$. Suppose

- Efstathios will test $H_0: \mu \leq 5$ versus $H_1: \mu > 5$ with
  
  $\text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - 5)/2 > z_{0.10}$

- Dimitris will test $H_0: \mu \geq 5$ versus $H_1: \mu < 5$ with
  
  $\text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - 5)/2 < -z_{0.10}$

- Phoebe will test $H_0: \mu = 5$ versus $H_1: \mu \neq 5$ with
  
  $\text{Reject } H_0 \text{ iff } |\sqrt{n}(\bar{X}_n - 5)/2| > z_{0.05}$

Each will collect a sample of size $n = 20$.

1. If in truth $\mu = 4.5$, who may commit a Type I error?
2. If in truth $\mu = 5.5$, who may commit a Type I error?
3. If in truth $\mu = 4.5$, who may commit a Type II error?
4. If in truth $\mu = 5.5$, who may commit a Type II error?
Exercise: Let $X_1, \ldots, X_n \sim \text{Normal}(\mu, 4)$. Suppose

- Efstathios will test $H_0: \mu \leq 5$ versus $H_1: \mu > 5$ with
  \[ \text{Reject } H_0 \text{ iff } \sqrt{n} (\bar{X}_n - 5)/2 > z_{0.10} \]

- Dimitris will test $H_0: \mu \geq 5$ versus $H_1: \mu < 5$ with
  \[ \text{Reject } H_0 \text{ iff } \sqrt{n} (\bar{X}_n - 5)/2 < -z_{0.10} \]

- Phoebe will test $H_0: \mu = 5$ versus $H_1: \mu \neq 5$ with
  \[ \text{Reject } H_0 \text{ iff } |\sqrt{n} (\bar{X}_n - 5)/2| > z_{0.05} \]

Each will collect a sample of size $n = 20$.

6. If in truth $\mu = 4.5$, who is most likely to reject $H_0$?

6. If in truth $\mu = 4.5$, who is least likely to reject $H_0$?

7. Compute the power of each researcher’s test when $\mu = 4.5$.

8. Plot the power curves of the three researchers’ tests together.
Inference about the mean of a Normal population

Variance known

Sample mean densities

\[
\begin{array}{cccccc}
4.5 & 5.0 & 4.5 & 5.0 & 5.5 & 6.0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
|       |       |       |       |       |       |
0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
\end{array}
\]

\[
\begin{array}{c}
mu
\end{array}
\]

Power

\[
\begin{array}{cccccc}
4.5 & 5.0 & 5.5 & 6.0 & 6.5 & 7.0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
|       |       |       |       |       |       |
0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
\end{array}
\]

\[
\begin{array}{c}
mu
\end{array}
\]

\[
\begin{array}{c}
0.008
\end{array}
\]
Inference about the mean of a Normal population

Variance known

sample mean densities

\[
\begin{align*}
4.5 & \quad 5.0 \\
3.5 & \quad 4.0 \\
4.5 & \quad 5.0 \\
5.5 & 
\end{align*}
\]

power

\[
\begin{align*}
3.5 & \quad 4.0 \\
4.5 & \quad 5.0 \\
5.0 & \quad 5.5 
\end{align*}
\]

power 0.435
Inference about the mean of a Normal population

Variance known

sample mean
densities

0.0 0.2 0.4 0.6 0.8 1.0

mu

power

0.302

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STAT 513 fa 2020 Lec 02 slides
Tests about $\mu$ when $\sigma^2$ is unknown: Use $S_n$ instead of $\sigma$.

1. **Right-tailed test**: Test $H_0: \mu \leq \mu_0$ versus $H_1: \mu > \mu_0$ with the test
   
   Reject $H_0$ iff $\frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} > C_1$.

2. **Left-tailed test**: Test $H_0: \mu \geq \mu_0$ versus $H_1: \mu < \mu_0$ with the test
   
   Reject $H_0$ iff $\frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} < -C_1$.

3. **Two-sided test**: Test $H_0: \mu = \mu_0$ versus $H_1: \mu \neq \mu_0$ with the test
   
   Reject $H_0$ iff $\left| \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \right| > C_2$.

**Exercise**: For any $\alpha \in (0, 1)$, find $C_1$ and $C_2$ such that these have size $\alpha$. 
Exercise: Let $X_1, \ldots, X_{20} \sim \text{Normal}(\mu, \sigma^2)$.

1. Give a test of $H_0: \mu \geq -1$ versus $H_1: \mu < -1$ which will make a Type I error with probability no greater than $\alpha = 0.01$.

2. Do the same as in part (i), but supposing that $\sigma$ is known.

3. Discuss why the critical values are different and which test has greater power.

Power of the $t$-tests is more complicated to compute than that of the $Z$-tests...
Non-central $t$-distribution

Let $Z \sim \text{Normal}(0, 1)$ and $W \sim \chi^2_\nu$ be independent rvs and let $\phi$ be a constant.

Then the distribution of the rv

$$T = \frac{Z + \phi}{\sqrt{W/\nu}}$$

is called the **non-central $t$-distribution** with df $\nu$ and **non-centrality parameter** $\phi$.

Denote the distribution by $t_{\phi,\nu}$. When $\phi = 0$, the non-central $t$-distributions are the same as the $t$-distributions.

For $\xi \in (0, 1)$, let $t_{\phi,\nu,\xi}$ be value s.t. $\xi = P(T > t_{\phi,\nu,\xi})$ when $T \sim t_{\phi,\nu}$.

Let $F_{t_{\phi,\nu}}$ denote the cdf of the non-central $t$-dist. with df $\nu$ and ncp $\phi$. 
Non-central $t$-distribution pivot quantity result

If $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, then

$$\sqrt{n}(\bar{X}_n - \mu_0)/S_n \sim t_{\phi, n-1},$$

with $\phi = \sqrt{n}(\mu - \mu_0)/\sigma$.

Exercise: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$ and consider

$$H_0: \mu \leq \mu_0 \text{ versus } H_1: \mu > \mu_0.$$ 

Get an expression for the power function of the test

Reject $H_0$ iff

$$\frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} > t_{n-1, \alpha}.$$
Tests about $\mu$ when $\sigma^2$ unknown: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, $\sigma^2$ unknown.

For some null value $\mu_0$, define the test statistic

$$T_n = \sqrt{n}(\bar{X}_n - \mu_0)/S_n.$$ 

Then we have the following:

<table>
<thead>
<tr>
<th>$H_0$</th>
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</thead>
<tbody>
<tr>
<td>$\mu \leq \mu_0$</td>
<td>$\mu &gt; \mu_0$</td>
<td>$T_n &gt; t_{n-1,\alpha}$</td>
<td>$1 - F_{t_{\phi,n-1}}(t_{n-1,\alpha})$</td>
</tr>
<tr>
<td>$\mu \geq \mu_0$</td>
<td>$\mu &lt; \mu_0$</td>
<td>$T_n &lt; -t_{n-1,\alpha}$</td>
<td>$F_{t_{\phi,n-1}}(-t_{n-1,\alpha})$</td>
</tr>
<tr>
<td>$\mu = \mu_0$</td>
<td>$\mu \neq \mu_0$</td>
<td>$</td>
<td>T_n</td>
</tr>
</tbody>
</table>

In the above, $\phi = \sqrt{n}(\mu - \mu_0)/\sigma$. 

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Exercise: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$. Suppose

- Efstatheios will test $H_0: \mu \leq 5$ vs $H_1: \mu > 5$ with

  \[
  \text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - 5)/S_n > t_{n-1,0.10}
  \]

- Dimitris will test $H_0: \mu \geq 5$ vs $H_1: \mu < 5$ with

  \[
  \text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - 5)/S_n < -t_{n-1,0.10}
  \]

- Phoebe will test $H_0: \mu = 5$ vs $H_1: \mu \neq 5$ with

  \[
  \text{Reject } H_0 \text{ iff } |\sqrt{n}(\bar{X}_n - 5)/2|/S_n > t_{n-1,0.05}
  \]

Each will collect a sample of size $n = 20$.

1. What is the size of each test?
2. Compute the power of each researcher’s test when $\mu = 4.5$ and $\sigma^2 = 4$.
3. Plot the power curves of the three researchers’ tests together when $\sigma^2 = 4$. Then overlay the $\sigma^2$-known versions.
4. What can be said about the power curves?
Inference about the mean of a Normal population
Variance unknown
Exercise: Let $X_1, \ldots, X_{15}$ be heights of randomly selected 2-yr-old trees; assume they are Normal. Researchers will test $H_0: \sigma \geq 3$ versus $H_1: \sigma < 3$ with

Reject $H_0$ iff $S_{15} < 2$.

1. If in truth $\sigma = 2.5$, with what probability will the test reject $H_0$?
2. Find an expression for the power $\gamma(\sigma)$ of the test for any value of $\sigma$.
3. Over all $\sigma > 0$ find the maximum probability of a Type I error.
4. Make a plot of the power function $\gamma(\sigma)$ of the test over $\sigma \in (1, 4)$. 
Inference about the variance of a Normal population

![Graph showing the relationship between sigma and power.](image)
Tests about $\sigma^2$:

- **Right-tailed test**: Test $H_0: \sigma^2 \leq \sigma_0^2$ versus $H_1: \sigma^2 > \sigma_0^2$ with the test
  
  $\frac{(n-1)S_n^2}{\sigma_0^2} > C_{1r}$.

- **Left-tailed test**: Test $H_0: \sigma^2 \geq \sigma_0^2$ versus $H_1: \sigma^2 < \sigma_0^2$ with the test
  
  $\frac{(n-1)S_n^2}{\sigma_0^2} < C_{1l}$.

- **Two-sided test**: Test $H_0: \sigma^2 = \sigma_0^2$ versus $H_1: \sigma^2 \neq \sigma_0^2$ with the test
  
  $\frac{(n-1)S_n^2}{\sigma_0^2} < C_{2l}$ or $\frac{(n-1)S_n^2}{\sigma_0^2} > C_{2r}$.
Exercise: For any $\alpha \in (0, 1)$:

1. Find values $C_{1r}$, $C_{1l}$, and $C_{2l}$ and $C_{2r}$ such that these tests have size $\alpha$.
2. Find expressions for the power functions of the size-$\alpha$ tests.
Tests about $\sigma^2$: Let $X_1, \ldots, X_n \sim \text{Normal}(\mu, \sigma^2)$.

For some null value $\sigma_0^2$, define the test statistic

$$W_n = (n - 1) S_n^2 / \sigma_0^2.$$

Then we have the following:

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<th>Reject $H_0$ at $\alpha$ iff</th>
<th>Power function $\gamma(\sigma^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma^2 \leq \sigma_0^2$</td>
<td>$\sigma^2 &gt; \sigma_0^2$</td>
<td>$W_n &gt; \chi_{n-1,\alpha}^2$</td>
<td>$1 - F_{\chi_{n-1}^2}(\chi_{n-1,\alpha}^2(\sigma_0^2/\sigma^2))$</td>
</tr>
<tr>
<td>$\sigma^2 \geq \sigma_0^2$</td>
<td>$\sigma^2 &lt; \sigma_0^2$</td>
<td>$W_n &lt; \chi_{n-1,1-\alpha}^2$</td>
<td>$F_{\chi_{n-1}^2}(\chi_{n-1,1-\alpha}^2(\sigma_0^2/\sigma^2))$</td>
</tr>
<tr>
<td>$\sigma^2 = \sigma_0^2$</td>
<td>$\sigma^2 \neq \sigma_0^2$</td>
<td>$W_n &lt; \chi_{n-1,1-\alpha/2}^2$ or $W_n &gt; \chi_{n-1,\alpha/2}^2$</td>
<td>$F_{\chi_{n-1}^2}(\chi_{n-1,1-\alpha/2}^2(\sigma_0^2/\sigma^2)) + 1 - F_{\chi_{n-1}^2}(\chi_{n-1,1-\alpha/2}^2(\sigma_0^2/\sigma^2))$</td>
</tr>
</tbody>
</table>

In the above $F_{\chi_{n-1}^2}$ is the cdf of the $\chi_{n-1}^2$-distribution.
Exercise: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$ and

$$H_0: \sigma^2 = 2 \text{ versus } H_1: \sigma^2 \neq 2.$$ 

Plot the power as a function of $\sigma$ of the two-sided test with the size $\alpha = 0.01$ under the sample sizes $n = 5, 10, 20$. 
Equivalence between CIs and two-sided tests for the mean:

If \( X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2) \), a size-\( \alpha \) test of \( H_0: \mu = \mu_0 \) vs \( H_1: \mu \neq \mu_0 \) is

\[
\text{Reject } H_0 \text{ iff } \left| \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \right| > t_{n-1, \alpha/2}.
\]

Recall that a \((1 - \alpha) \times 100\%\) confidence interval for \( \mu \) is given by

\[
\bar{X}_n \pm t_{n-1, \alpha/2} S_n/\sqrt{n}.
\]

Equivalent test is to check whether \( \mu_0 \) contained in CI:

\[
\text{Reject } H_0 \text{ iff } \mu_0 \notin (\bar{X}_n - t_{n-1, \alpha/2} S_n/\sqrt{n}, \bar{X}_n + t_{n-1, \alpha/2} S_n/\sqrt{n}).
\]
Connection between two-sided tests and confidence intervals

Equivalence between CIs and two-sided tests for the variance:

If \( X_1, \ldots, X_n \ \text{iid} \ \text{Normal}(\mu, \sigma^2) \), a size-\( \alpha \) test of \( H_0: \sigma^2 = \sigma_0^2 \) vs \( H_1: \sigma^2 \neq \sigma_0^2 \) is

\[
\text{Reject } H_0 \text{ iff } \frac{(n - 1)S_n^2}{\sigma_0^2} < \chi^2_{n-1,1-\alpha/2} \text{ or } \frac{(n - 1)S_n^2}{\sigma_0^2} > \chi^2_{n-1,\alpha/2}.
\]

Recall that a \((1 - \alpha) \times 100\%\) confidence interval for \( \mu \) is given by

\[
\left( \frac{(n - 1)S_n^2}{\chi^2_{n-1,\alpha/2}}, \frac{(n - 1)S_n^2}{\chi^2_{n-1,1-\alpha/2}} \right)
\]

Equivalent test is to check whether \( \sigma_0^2 \) contained in CI:

\[
\text{Reject } H_0 \text{ iff } \sigma_0^2 \notin \left( \frac{(n - 1)S_n^2}{\chi^2_{n-1,\alpha/2}}, \frac{(n - 1)S_n^2}{\chi^2_{n-1,1-\alpha/2}} \right).
\]