STAT 513 fa 2020 Lec 03 slides

Measuring strength of evidence against the null with p-values

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Discuss: Consider the case of Vinaya and her younger brother Anuj, who wish to test H_0 vs H_1. Each gathers data, and

- Anuj rejects H_0 based on a test which has size 0.10 and
- Vinaya rejects H_0 based on a test which has size 0.01.

Whose result is more "significant"?

Significance level of a test

If the size of a test is less than or equal to α, we will say that the test has *significance level* α.

At what significance levels would the observed data lead to a rejection of H_0?

This is a way to measure the strength of observed evidence against H_0.

The p-value

The smallest significance level α at which the observed data would lead to a rejection of H_0 is called the *p-value*.

Exercise: Let $X_1, \dots, X_{10} \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$ and consider

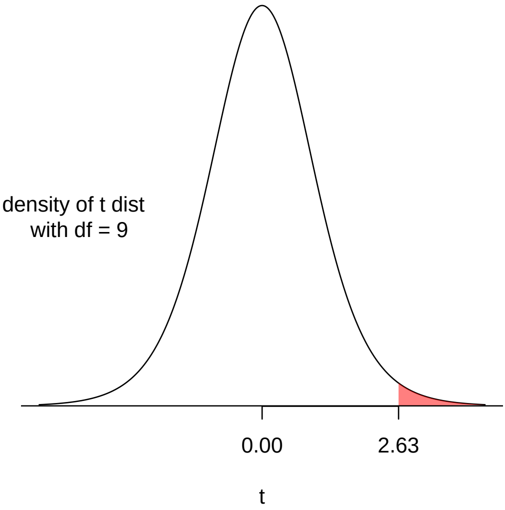
$$H_0\colon \mu \leq 5 \text{ versus } H_1\colon \mu > 5.$$

Suppose $\sqrt{10}(\bar{X}_{10} - 5)/S_{10} = 2.63$.

1. What is our decision about H_0 versus H_1 using significance level $\alpha = 0.05$?
2. What is our decision about H_0 versus H_1 using significance level $\alpha = 0.01$?
3. Compute the size of the test

$$\text{Reject } H_0 \text{ iff } \sqrt{10}(\bar{X}_{10} - 5)/S_{10} > 2.63.$$

4. What is the smallest significance level at which the observed random sample, for which $\sqrt{10}(\bar{X}_{10} - 5)/S_{10} = 2.63$, would lead to a rejection of H_0?
5. Draw a plot of the density of the test statistic when $\mu = 5$ and shade the area corresponding to the p-value.

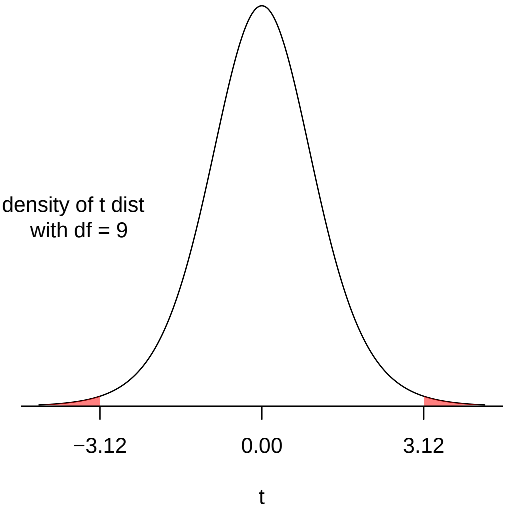

density of t dist
with df = 9
0.00
2.63
t

Exercise: Let $X_1, \dots, X_{10}$ be a random sample from the Normal(μ, σ^2) distribution, where μ and σ^2 are unknown, and suppose we wish to test

$$H_0\colon \mu = 8 \text{ versus } H_1\colon \mu \neq 8.$$

Suppose $\sqrt{10}(\bar{X}_{10} - 8)/S_{10} = -3.12$.

1. What is our decision about H_0 versus H_1 using significance level $\alpha = 0.05$?
2. What is our decision about H_0 versus H_1 using significance level $\alpha = 0.01$?
3. If H_0 is true, what is the probability of getting a sample which carries as much or more evidence against H_0?
4. Draw a plot of the density of the test statistic when $\mu = 8$ and shade the area corresponding to the p-value.


density of t dist
with df = 9
−3.12
0.00
3.12
t

Tests about μ: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$.

For some null value μ_0, define the test statistic

$$T_n = \sqrt{n}(\bar{X}_n - \mu_0)/S_n.$$

Then we have the following p-value formulas:

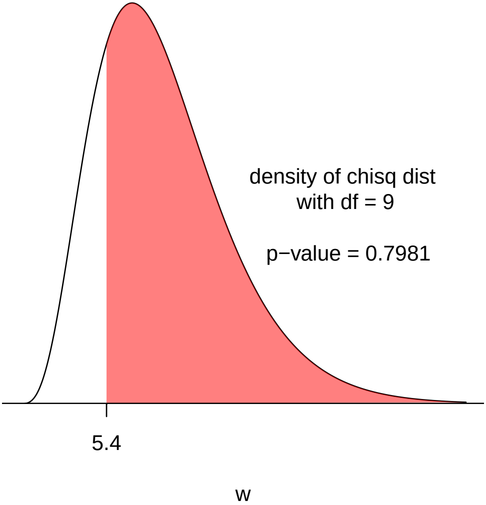
H_0	H_1	Reject H_0 iff	p-value
$\mu \leq \mu_0$	$\mu > \mu_0$	$T_n > t_{n-1,\alpha}$	$1 - F_{t_{n-1}}(T_n)$
$\mu \geq \mu_0$	$\mu < \mu_0$	$T_n < -t_{n-1,\alpha}$	$F_{t_{n-1}}(T_n)$
$\mu = \mu_0$	$\mu \neq \mu_0$	$\lvert T_n\rvert > t_{n-1,\alpha/2}$	$2(1 - F_{t_{n-1}}(\lvert T_n\rvert))$

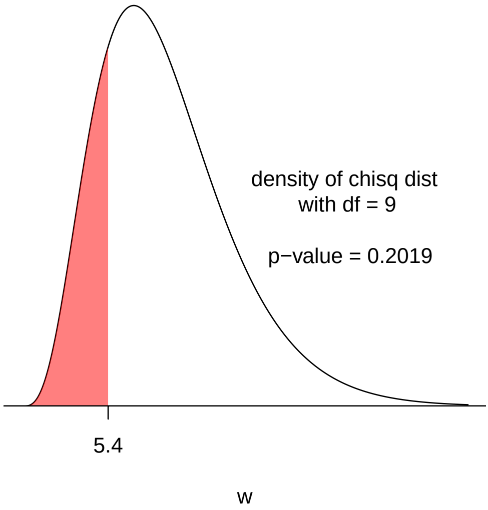
In the above, $F_{t_{n-1}}$ is the cdf of the t_{n-1} distribution.

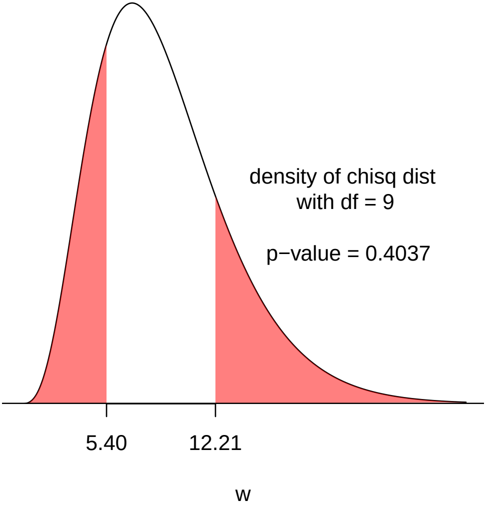
Exercise: Let $X_1, \dots, X_{10} \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$. Suppose $S_{10}^2 = 3$.

Give the p-values for testing the following sets of hypotheses:

1. H_0: $\sigma^2 \leq 5$ versus H_1: $\sigma^2 > 5$
2. H_0: $\sigma^2 \geq 5$ versus H_1: $\sigma^2 < 5$
3. H_0: $\sigma^2 = 5$ versus H_1: $\sigma^2 \neq 5$


density of chisq dist
with df = 9
p−value = 0.7981
5.4
w


density of chisq dist
with df = 9
p−value = 0.2019
5.4
w


density of chisq dist
with df = 9
p−value = 0.4037
5.40
12.21
w

Tests about σ^2: Let $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$.

For some null value σ_0^2, define the test statistic

$$W_n = (n-1)S_n^2/\sigma_0^2.$$

Then we have the following p-value formulas:

H_0	H_1	Reject H_0 at α iff	p-value
$\sigma^2 \leq \sigma_0^2$	$\sigma^2 > \sigma_0^2$	$W_n > \chi^2_{n-1,\alpha}$	$1 - F_{\chi^2_{n-1}}(W_n)$
$\sigma^2 \geq \sigma_0^2$	$\sigma^2 < \sigma_0^2$	$W_n < \chi^2_{n-1,1-\alpha}$	$F_{\chi^2_{n-1}}(W_n)$
$\sigma^2 = \sigma_0^2$	$\sigma^2 \neq \sigma_0^2$	$W_n < \chi^2_{n-1,1-\alpha/2}$ or $W_n > \chi^2_{n-1,\alpha/2}$	$2 \cdot \min\{F_{\chi^2_{n-1}}(W_n), 1 - F_{\chi^2_{n-1}}(W_n)\}$

In the above, $F_{\chi^2_{n-1}}$ is the cdf of the χ^2_{n-1} distribution.