# STAT 513 fa 2020 Lec 04 slides

# Comparing two Normal populations

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

- Consider two random samples of sizes $n_1$ and $n_2$:

$$X_{11}, \dots, X_{1n_1} \overset{\text{ind}}{\sim} \text{Normal}(\mu_1, \sigma_1^2)$$
$$X_{21}, \dots, X_{2n_2} \overset{\text{ind}}{\sim} \text{Normal}(\mu_2, \sigma_2^2),$$

  where $\mu_1$, $\mu_2$, $\sigma_1^2$, and $\sigma_2^2$ are unknown.
- Let
  - $\bar{X}_1$ and $S_1^2$ denote the mean and variance of the first sample
  - $\bar{X}_2$ and $S_2^2$ denote the mean and variance of the second sample.
- Wish to compare
  - $\mu_1$ and $\mu_2$ by testing hypotheses about the difference $\mu_1 - \mu_2$
  - $\sigma_1^2$ and $\sigma_2^2$ by testing hypotheses about the ratio $\sigma_1^2/\sigma_2^2$

## Tests about $\mu_1 - \mu_2$ when $\sigma_1^2 = \sigma_2^2$:

1. *Right-tailed test*: Test $H_0$: $\mu_1 - \mu_2 \leq \delta_0$ versus $H_1$: $\mu_1 - \mu_2 > \delta_0$ with the test

$$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_1 - \bar{X}_2 - \delta_0}{S_{\text{pooled}}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} > C_1.$$

2. *Left-tailed test*: Test $H_0$: $\mu_1 - \mu_2 \geq \delta_0$ versus $H_1$: $\mu_1 - \mu_2 < \delta_0$ with the test

$$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_1 - \bar{X}_2 - \delta_0}{S_{\text{pooled}}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} < -C_1.$$

3. *Two-sided test*: Test $H_0$: $\mu_1 - \mu_2 = \delta_0$ versus $H_1$: $\mu_1 - \mu_2 \neq \delta_0$ with the test

$$\text{Reject } H_0 \text{ iff } \left|\frac{\bar{X}_1 - \bar{X}_2 - \delta_0}{S_{\text{pooled}}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\right| > C_2.$$

**Exercise:** For any $\alpha \in (0,1)$:

1. Find $C_1$ and $C_2$ such that the tests on the previous slide have size $\alpha$.
2. Find the power functions, in terms of $\delta = \mu_1 - \mu_2$, of the size-$\alpha$ tests.

## Distribution of equal-variances two-sample t-statistic

Let

$$X_{11}, \dots, X_{1n_1} \overset{\text{ind}}{\sim} \text{Normal}(\mu_1, \sigma_1^2)$$

$$X_{21}, \dots, X_{2n_2} \overset{\text{ind}}{\sim} \text{Normal}(\mu_2, \sigma_2^2),$$

with $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Suppose $\mu_1 - \mu_2 = \delta$. Then for any $\delta_0$ we have

$$\frac{\bar{X}_1 - \bar{X}_2 - \delta_0}{S_{\text{pooled}}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{\phi, n_1+n_2-2}, \quad \text{where } \phi = \frac{\delta - \delta_0}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

Show above...

Tests about $\mu_1 - \mu_2$: Let $X_{k1}, \dots, X_{kn_k} \overset{\text{ind}}{\sim} \text{Normal}(\mu_k, \sigma^2)$, $k = 1, 2$.

For some null value $\delta_0$, define the test statistic

$$T = \frac{\bar{X}_1 - \bar{X}_2 - \delta_0}{S_{\text{pooled}}\sqrt{1/n_1 + 1/n_2}}.$$

Then we have the following:

| $H_1$ | Reject $H_0$ at $\alpha$ iff | Power function $\gamma(\mu)$ |
|---|---|---|
| $\mu_1 - \mu_2 > \delta_0$ | $T > t_{\nu,\alpha}$ | $1 - F_{t_{\phi,\nu}}(t_{\nu,\alpha})$ |
| $\mu_1 - \mu_2 < \delta_0$ | $T < -t_{\nu,\alpha}$ | $F_{t_{\phi,\nu}}(-t_{\nu,\alpha})$ |
| $\mu_1 - \mu_2 \neq \delta_0$ | $\lvert T\rvert > t_{\nu,\alpha/2}$ | $1 - [F_{t_{\phi,\nu}}(t_{\nu,\alpha/2}) - F_{t_{\phi,\nu}}(-t_{\nu,\alpha/2})]$ |

where $\nu = n_1 + n_2 - 2$ and $\phi = (\delta - \delta_0)/(\sigma\sqrt{1/n_1 + 1/n_2})$.

**Exercise:** Let $X_{k1}, \dots, X_{kn_k} \overset{\text{ind}}{\sim} \text{Normal}(\mu_k, \sigma_k^2)$, $k = 1, 2$, with $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Suppose:

- Simone to test $H_0$: $\mu_1 - \mu_2 \leq 0$ vs $H_1$: $\mu_1 - \mu_2 > 0$ with

$$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_1 - \bar{X}_2}{S_{\text{pooled}}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} > t_{n_1+n_2-2,0.10}$$

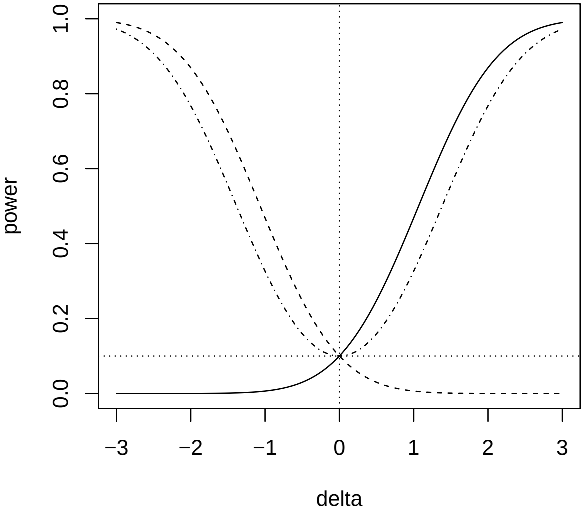
- Helmut to test $H_0$: $\mu_1 - \mu_2 \geq 0$ vs $H_1$: $\mu_1 - \mu_2 < 0$ with

$$\text{Reject } H_0 \text{ iff } \frac{\bar{X}_1 - \bar{X}_2}{S_{\text{pooled}}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} < -t_{n_1+n_2-2,0.10}$$

- Mechthild to test $H_0$: $\mu_1 - \mu_2 = 0$ vs $H_1$: $\mu_1 - \mu_2 \neq 0$ with

$$\text{Reject } H_0 \text{ iff } \left|\frac{\bar{X}_1 - \bar{X}_2}{S_{\text{pooled}}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\right| > t_{n_1+n_2-2,0.05}$$

1. What is the size of each test?
2. Plot the three power curves $\gamma(\delta)$ under $n_1 = 10$, $n_2 = 15$, and $\sigma^2 = 4$.


power
delta
0.0
0.2
0.4
0.6
0.8
1.0
−3
−2
−1
0
1
2
3

For the unequal variances $\sigma_1^2 \neq \sigma_2^2$ case, see Lecture 4 notes.

## Tests about $\sigma_1^2/\sigma_2^2$:

1. *Right-tailed test*: Test $H_0$: $\sigma_1^2/\sigma_2^2 \leq \vartheta_0$ versus $H_1$: $\sigma_1^2/\sigma_2^2 > \vartheta_0$ with the test

$$\text{Reject } H_0 \text{ iff } \frac{S_1^2}{S_2^2}/\vartheta_0 > C_{1r}.$$

2. *Left-tailed test*: Test $H_0$: $\sigma_1^2/\sigma_2^2 \geq \vartheta_0$ versus $H_1$: $\sigma_1^2/\sigma_2^2 < \vartheta_0$ with the test

$$\text{Reject } H_0 \text{ iff } \frac{S_1^2}{S_2^2}/\vartheta_0 < C_{1l}.$$

3. *Two-sided test*: Test $H_0$: $\sigma_1^2/\sigma_2^2 = \vartheta_0$ versus $H_1$: $\sigma_1^2/\sigma_2^2 \neq \vartheta_0$ with the test

$$\text{Reject } H_0 \text{ iff } \frac{S_1^2}{S_2^2}/\vartheta_0 < C_{2l} \text{ or } \frac{S_1^2}{S_2^2}/\vartheta_0 > C_{2r}.$$

**Exercise:** For any $\alpha \in (0,1)$:

1. Find $C_{1r}$, $C_{1l}$, and $C_{2l}$ and $C_{2r}$ such that these tests have size $\alpha$.
2. Find the power functions, in terms of $\vartheta = \sigma_1^2/\sigma_2^2$, of the size-$\alpha$ tests.

## Distribution of variance-ratio $F$-statistic

Let $X_{k1}, \dots, X_{kn_k} \stackrel{\text{ind}}{\sim} \text{Normal}(\mu_k, \sigma_k^2)$, $k = 1, 2$ with $\sigma_1^2/\sigma_2^2 = \vartheta$. Then

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2}{S_2^2}/\vartheta \sim F_{n_1-1, n_2-1}.$$

$F_{n_1-1,n_2-1}$ denotes $F$-dist. with numerator df $n_1 - 1$ and denominator df $n_2 - 1$.

Tests about $\sigma_1^2/\sigma_2^2$: Let $X_{k1}, \dots, X_{kn_k} \overset{\text{ind}}{\sim} \text{Normal}(\mu_k, \sigma_k^2)$, $k = 1, 2$.

For some null value $\vartheta_0$, define the test statistic

$$R = (S_1/S_2)/\vartheta_0.$$

Then we have the following:

| $H_1$ | Reject $H_0$ at $\alpha$ iff | Power function $\gamma(\vartheta)$ |
|---|---|---|
| $\sigma_1^2/\sigma_2^2 > \vartheta_0$ | $R > F_{n_1-1,n_2-1,\alpha}$ | $1 - F_{F_{n_1-1,n_2-1}}(F_{n_1-1,n_2-1,\alpha}(\vartheta_0/\vartheta))$ |
| $\sigma_1^2/\sigma_2^2 < \vartheta_0$ | $R < F_{n_1-1,n_2-1,1-\alpha}$ | $F_{F_{n_1-1,n_2-1}}(F_{n_1-1,n_2-1,1-\alpha}(\vartheta_0/\vartheta))$ |
| $\sigma_1^2/\sigma_2^2 \neq \vartheta_0$ | $R < F_{n_1-1,n_2-1,1-\alpha/2}$ or $R > F_{n_1-1,n_2-1,\alpha/2}$ | $F_{F_{n_1-1,n_2-1}}(F_{n_1-1,n_2-1,1-\alpha/2}(\vartheta_0/\vartheta))$ $+ 1 - F_{F_{n_1-1,n_2-1}}(F_{n_1-1,n_2-1,\alpha/2}(\vartheta_0/\vartheta))$ |

**Exercise:** Let $X_{k1}, \dots, X_{kn_k} \stackrel{\text{ind}}{\sim} \text{Normal}(\mu_k, \sigma_k^2)$, $k = 1, 2$. Suppose:

- Kwame to test $H_0$: $\sigma_1^2/\sigma_2^2 \leq 1$ vs $H_1$: $\sigma_1^2/\sigma_2^2 > 1$ with

$$\text{Rej. } H_0 \text{ iff } S_1^2/S_2^2 > F_{n_1-1,n_2-1,0.10}$$

- Ama to test $H_0$: $\sigma_1^2/\sigma_2^2 \geq 1$ vs $H_1$: $\sigma_1^2/\sigma_2^2 < 1$ with

$$\text{Rej. } H_0 \text{ iff } S_1^2/S_2^2 < F_{n_1-1,n_2-1,0.90}$$

- Kobe to test $H_0$: $\sigma_1^2/\sigma_2^2 = 1$ vs $H_1$: $\sigma_1^2/\sigma_2^2 \neq 1$ with

$$\text{Rej. } H_0 \text{ iff } S_1^2/S_2^2 < F_{n_1-1,n_2-1,0.95} \text{ or } S_1^2/S_2^2 > F_{n_1-1,n_2-1,0.05}$$

1. Suppose $\sigma_1^2 = 1$ and $\sigma_2^2 = 2$. Who may commit a Type I error?
2. Suppose $\sigma_1^2 = 2$ and $\sigma_2^2 = 1$. Who may commit a Type I error?
3. Suppose $\sigma_1^2 = 1$ and $\sigma_2^2 = 2$. Who may commit a Type II error?
4. Suppose $\sigma_1^2 = 2$ and $\sigma_2^2 = 1$. Who may commit a Type II error?

**Exercise:** Let $X_{k1}, \dots, X_{kn_k} \stackrel{\text{ind}}{\sim} \text{Normal}(\mu_k, \sigma_k^2)$, $k = 1, 2$. Suppose:

- Kwame to test $H_0$: $\sigma_1^2/\sigma_2^2 \leq 1$ vs $H_1$: $\sigma_1^2/\sigma_2^2 > 1$ with

$$\text{Rej. } H_0 \text{ iff } S_1^2/S_2^2 > F_{n_1-1, n_2-1, 0.10}$$

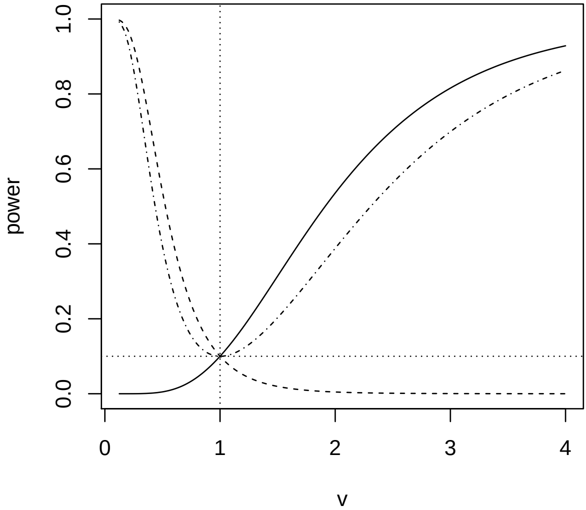
- Ama to test $H_0$: $\sigma_1^2/\sigma_2^2 \geq 1$ vs $H_1$: $\sigma_1^2/\sigma_2^2 < 1$ with

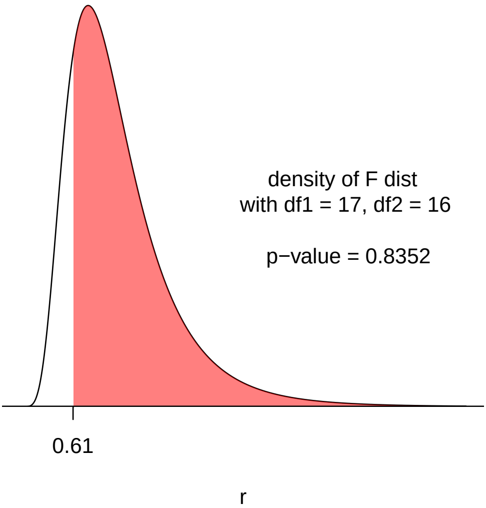
$$\text{Rej. } H_0 \text{ iff } S_1^2/S_2^2 < F_{n_1-1, n_2-1, 0.90}$$

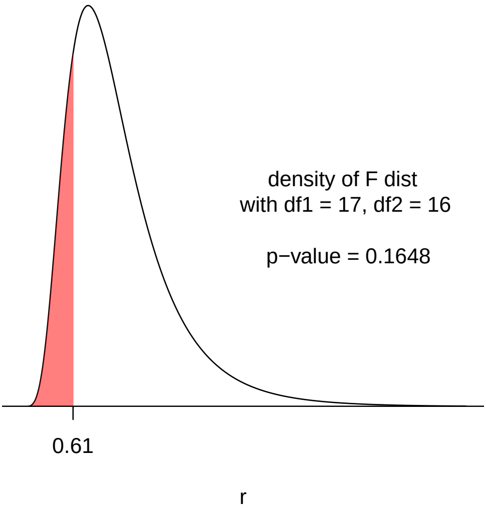
- Kobe to test $H_0$: $\sigma_1^2/\sigma_2^2 = 1$ vs $H_1$: $\sigma_1^2/\sigma_2^2 \neq 1$ with

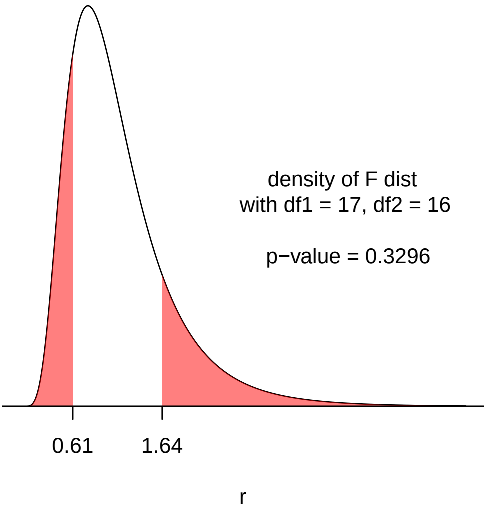
$$\text{Rej. } H_0 \text{ iff } S_1^2/S_2^2 < F_{n_1-1, n_2-1, 0.95} \text{ or } S_1^2/S_2^2 > F_{n_1-1, n_2-1, 0.05}$$

5. What is the size of each test?
6. Under $n_1 = 18$ and $n_2 = 17$, plot the three power curves $\gamma(\vartheta)$ together.
7. Suppose that with samples of sizes $n_1 = 18$ and $n_2 = 17$, $S_1^2 = 1.15$ and $S_2^2 = 1.87$ is observed. Compute the $p$-value of each researcher's test.


power
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
v


density of F dist
with df1 = 17, df2 = 16
p−value = 0.8352
0.61
r


density of F dist
with df1 = 17, df2 = 16
p−value = 0.1648
0.61
r


density of F dist
with df1 = 17, df2 = 16
p−value = 0.3296
0.61
1.64
r