STAT 513 fa 2018 Exam I (take-home)

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assigned: Tuesday, Oct 2nd, 2018
due: Thursday, Oct 4th, 2018

Instructions:

- Looking at course notes IS allowed.
- Working with others IS NOT allowed. Asking a friend to help you puts the friend in the uncomfortable position of wanting to be nice but not wanting to break a rule. I recommend not doing that to your friends, because friendships are more important than exam grades.
- Write solutions on blank sheets of paper and turn these sheets in WITH A BLANK SHEET ON TOP (to cover answers) which has only your name on it. You do not need to turn in a copy of the exam itself.
- I expect answers to by very neatly written, since you have the time. Partial credit will be given only for legible work.

The table below gives some values of the function $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$:

z	1.282	1.645	1.96	2.576
$\Phi(z)$	0.9	0.95	0.975	0.995

Let $X_1, \dots, X_n$ be a random sample with likelihood function $L(\theta; X_1, \dots, X_n)$. Then for hypotheses of the form H_0: $\theta = \theta_0$ versus H_1: $\theta \neq \theta_0$ the likelihood ratio takes the form

$$\text{LR}(X_1, \dots, X_n) = \frac{L(\theta_0; X_1, \dots, X_n)}{L(\hat{\theta}; X_1, \dots, X_n)},$$

where $\hat{\theta}$ is the maximum likelihood estimator of θ.

1. Let $X_1, \ldots, X_n$ be a random sample from the Normal$(\mu, 1)$ distribution, where μ is unknown. A researcher plans to test H_0: $\mu \geq 5$ versus H_1: $\mu < 5$ with the test

$$\text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - 5) < -1.96.$$

   (a) Give the size of the test.

   (b) Make a sketch which gives the shape of the power curve, indicating its height at $\mu = 5$.

   (c) Give an expression for the power $\gamma(\mu)$ at any value of μ.

   (d) Which of the following numbers, when rounded up, is equal to the smallest sample size under which the test will reject H_0 with probability at least 0.90 for all $\mu \leq 4$? *Hint: You must do some calculations involving the power function.* **No points will be awarded if you do not show your work, even if you select the right answer.**

      A. $(1.96 + 2.575)^2$
      B. $(1.96 + 2.575)^2/(5)^2$
      C. $(1.96 + 1.282)^2$
      D. $(2.575 + 2.575)^2/4$
      E. $(1.96 + 2.575)^2/(\sqrt{4})^2$
      F. $(1.96 + 1.96)^2/(\sqrt{5})^2$
      G. $2^2(1.96 + 1.282)^2$
      H. $(1.96 - 1.282)^2/5$
      I. $2^2(1.96)^2/5^2$
      J. $2^2(1.282)^2$

   (e) Assuming that the correct sample size from the previous part is used, sketch the power curve of the test; include vertical lines positioned at $\mu = 4$ and $\mu = 5$ and horizontal lines positioned at the heights 0.90 and 0.025.

   (f) Suppose that a random sample of size n results in $\sqrt{n}(\bar{X}_n - 5) = -2.65$. In which of the following intervals does the p-value lie? *Hint: Consult the table of values of the standard Normal cdf $\Phi(z)$.* **No points will be awarded if you do not show your reasoning, even if you select the right answer.**

      A. $(0, .005)$
      B. $[.005, .01)$
      C. $[.01, .05)$
      D. $[.05, .5)$
      E. $[.5, 1)$

   (g) Suppose that a random sample of size n results in $\sqrt{n}(\bar{X}_n - 5) = 2.65$. In which of the following intervals does the p-value lie? **No points will be awarded if you do not show your reasoning, even if you select the right answer.**

      A. $(0, .005)$
      B. $[.005, .01)$
      C. $[.01, .05)$
      D. $[.05, .5)$
      E. $[.5, 1)$

2. Let n^* be the smallest sample size required to detect a deviation as small as δ^* from the null with probability at least γ^* using a test with size α. Fill in the blanks:
   (a) For ____________ (larger/smaller) δ^* a ____________ (larger/smaller) n^* is required.
   (b) For ____________ (larger/smaller) γ^* a ____________ (larger/smaller) n^* is required.
   (c) For ____________ (larger/smaller) α a ____________ (larger/smaller) n^* is required.

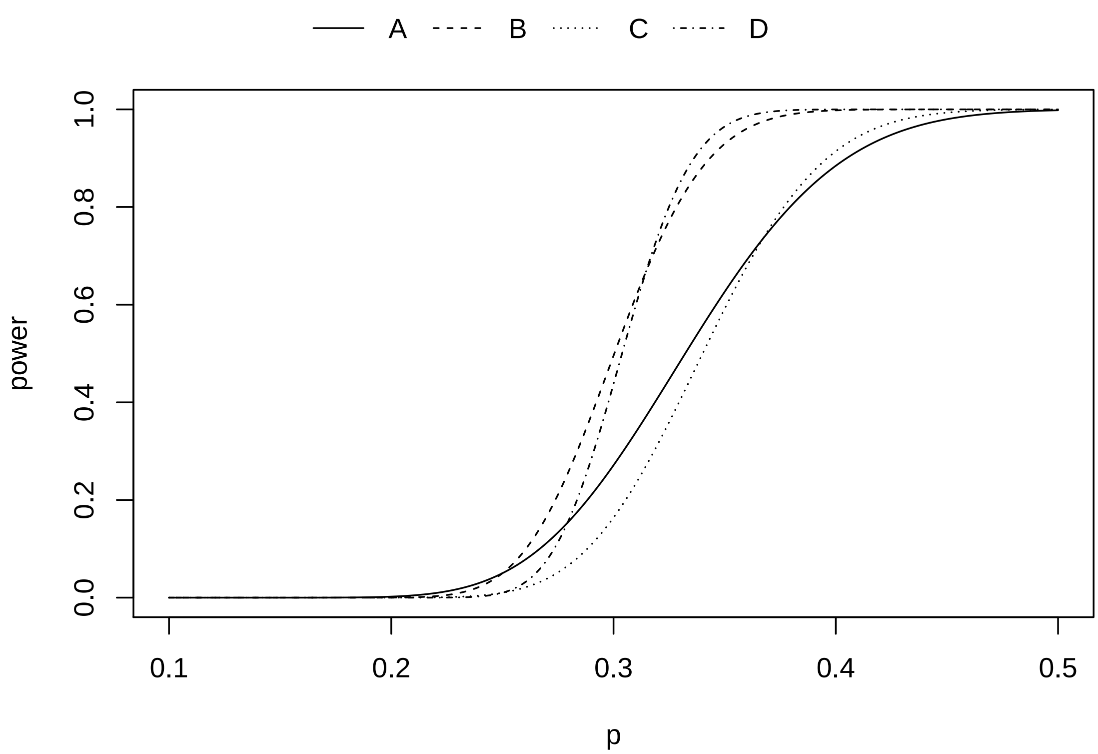
3. Let $X_1, \dots, X_n$ be a random sample from the Bernoulli(p) distribution, where p is unknown, and suppose it is of interest to test

$$H_0\colon\ p \leq 1/4 \text{ versus } p > 1/4.$$

Power curves of the test

$$\text{Reject } H_0 \text{ iff } \sqrt{n}(\hat{p}_n - 1/4)/\sqrt{1/4(1-1/4)} > z_\alpha$$

are shown under four settings, A, B, C, and D, which correspond to different values of α and n.



Answer the following by looking carefully at the power curves. **Explain answers to recieve credit!**

(a) Which settings have $\alpha = 0.05$?

(b) Between settings C and D, which has the smaller sample size n?

(c) Which of the four settings has the largest sample size n?

(d) Under which setting is the probability of a Type II error the smallest when $p = 0.28$?

4. Let $X_1, \dots, X_n$ be a random sample from the distribution with pdf given by

$$f(x) = \beta^{-1} \exp(-x\beta^{-1}) \mathbb{1}(x > 0),$$

where $\beta \geq 0$ is unknown, and suppose it is of interest to test the hypotheses

$$H_0\colon\ \beta = \beta_0 \text{ versus } H_1\colon\ \beta \neq \beta_0.$$

(a) Give the likelihood function $L(\beta; X_1, \dots, X_n)$ for $X_1, \dots, X_n$.

(b) Give the log-likelihood function $\ell(\beta; X_1, \dots, X_n)$ for $X_1, \dots, X_n$.

(c) Find the maximum likelihood estimator $\hat{\beta}$ of β based on $X_1, \dots, X_n$.

(d) Give the likelihood ratio $\text{LR}(X_1, \dots, X_n)$ for testing the hypotheses $H_0\colon\ \beta = \beta_0$ versus $H_1\colon\ \beta \neq \beta_0$.

(e) Show that the rejection criterion $\text{LR}(X_1, \dots, X_n) < c$ of the likelihood ratio test is equivalent to

$$\frac{\bar{X}_n}{\beta_0} \exp\left(-\frac{\bar{X}_n}{\beta_0}\right) < c^{1/n} e^{-1}$$

for any $c \in [0, 1]$.

(f) Since the function ze^{-z} is strictly increasing for $z < 1$ and strictly decreasing for $z > 1$, we have that rejecting H_0 when $\text{LR}(X_1, \dots, X_n) < c$ is equivalent to rejecting H_0 when

$$\bar{X}_n < c_1 \text{ or } \bar{X}_n > c_2$$

for some c_1 and c_2. Explain in words how we can use the fact that

$$X_1, \dots, X_n \overset{iid}{\sim} \text{Exponential}(\beta) \implies \bar{X}_n \sim \text{Gamma}(n, \beta/n)$$

to choose c_1 and c_2 such that the test has size α for any $\alpha \in (0, 1)$.