STAT 513 fa 2019 Exam I

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Do not open this test until told to do so; no calculators allowed; no notes allowed; no books allowed; show your work so that partial credit may be given.

The table below gives some values of the function $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$:

z	0.841	1.282	1.645	1.96	2.326	2.576
$\Phi(z)$	0.80	0.90	0.95	0.975	.990	0.995

Result: Let $X_1, \dots, X_n$ be a random sample from the Normal(μ, σ^2) distribution. Then

$$\frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \sim t_{\phi, n-1}, \text{ where } \phi = \frac{\mu - \mu_0}{\sigma/\sqrt{n}}.$$

1. Let $X_1, \ldots, X_n \stackrel{\text{ind}}{\sim} \text{Bernoulli}(p)$, where p is unknown, and suppose the hypotheses

$$H_0\colon\ p \leq 1/8 \text{ versus } H_1\colon\ p > 1/8$$

are to be tested using the test

$$\text{Reject } H_0 \text{ iff } \bar{X}_n > 1/8 + C\sqrt{(1/8)(7/8)/n}$$

for some $C > 0$, where $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$.

   (a) For a given sample size n, state the distribution of $\sum_{i=1}^n X_i$.

   (b) Write down an expression for the power of the test as a function of p. Write the function such that it can be evaluated using the cdf of $\sum_{i=1}^n X_i$.

   (c) Sketch the shape of the power curve.

   (d) Give a function of $\bar{X}_n$ which, assuming $p = 1/8$, will have a sampling distribution closer and closer to the Normal$(0, 1)$ distribution for larger and larger n. *Hint: Central Limit Theorem.*

   (e) Find the value of C such that the test given above will have size approaching 0.05 as $n \to \infty$.

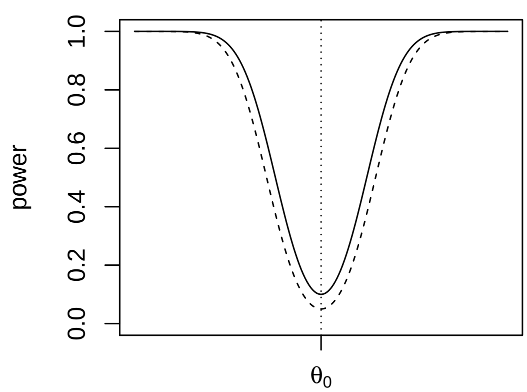
2. Consider a parameter $\theta \in \Theta$.

   (a) Consider testing $H_0\colon\ \theta \in \Theta_0$ versus $H_1\colon\ \theta \in \Theta_1$, and suppose that instead of collecting real data related to θ, you decide you will just flip a fair coin and use the following test:

   Reject H_0 iff the coin flip is "heads".

   What is the size of the test?

   (b) Suppose you wish to test $H_0\colon\ \theta = \theta_0$ versus $H_1\colon\ \theta \neq \theta_0$, and you must choose between the two tests which have the power curves plotted below.



   Discuss the relative merits of the two tests.

(c) Suppose you will test the hypotheses H_0: $\theta = \theta_0$ versus H_1: $\theta \neq \theta_0$ with the test

$$\text{Reject } H_0 \text{ iff } T < a \text{ or } T > b,$$

where T is some quantity you will compute on observed data and a and b are some real numbers. Suppose that when $\theta = \theta_0$, the quantity T has a distribution with cdf given by the function F.

i. Write down the probability of a Type I error in terms of the function F.

ii. How could you find values a and b such that the test has size no greater than $\alpha \in (0, 1)$?

3. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, σ^2 known, and consider H_0: $\mu \geq \mu_0$ versus H_1: $\mu < \mu_0$. The test

$$\text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - \mu_0)/\sigma < -z_\alpha$$

has power function given by

$$\gamma(\mu) = \Phi(-z_\alpha - \sqrt{n}(\mu - \mu_0)/\sigma).$$

(a) Carefully sketch the power curve, including a vertical line positioned at μ_0 and a horizontal line positioned at α.

(b) Suppose it is of interest to detect a deviation from the null of size σ (that is, you wish to detect whether $\mu < \mu_0 - \sigma$) with probability 0.80 while keeping the probability of a Type I error bounded by 0.01. Find the required sample size.

(c) Say whether the required sample size would increase or decrease for each of the following modifications to part (b).

i. If you kept the Type I error probability bounded by 0.05 instead of 0.01.

ii. If you wanted to detect the deviation with probability 0.90 instead of 0.80.

iii. If you wanted to detect a deviation from the null of size $\sigma/2$ instead of σ.

(d) Suppose you collect data such that $\sqrt{n}(\bar{X}_n - \mu_0)/\sigma = -1.58$. In which of the following intervals does the p-value lie?

A. $(0, .005)$

B. $[.005, .01)$

C. $[.01, .05)$

D. $[.05, .1)$

E. $[.1, .5)$

F. $[.5, 1)$

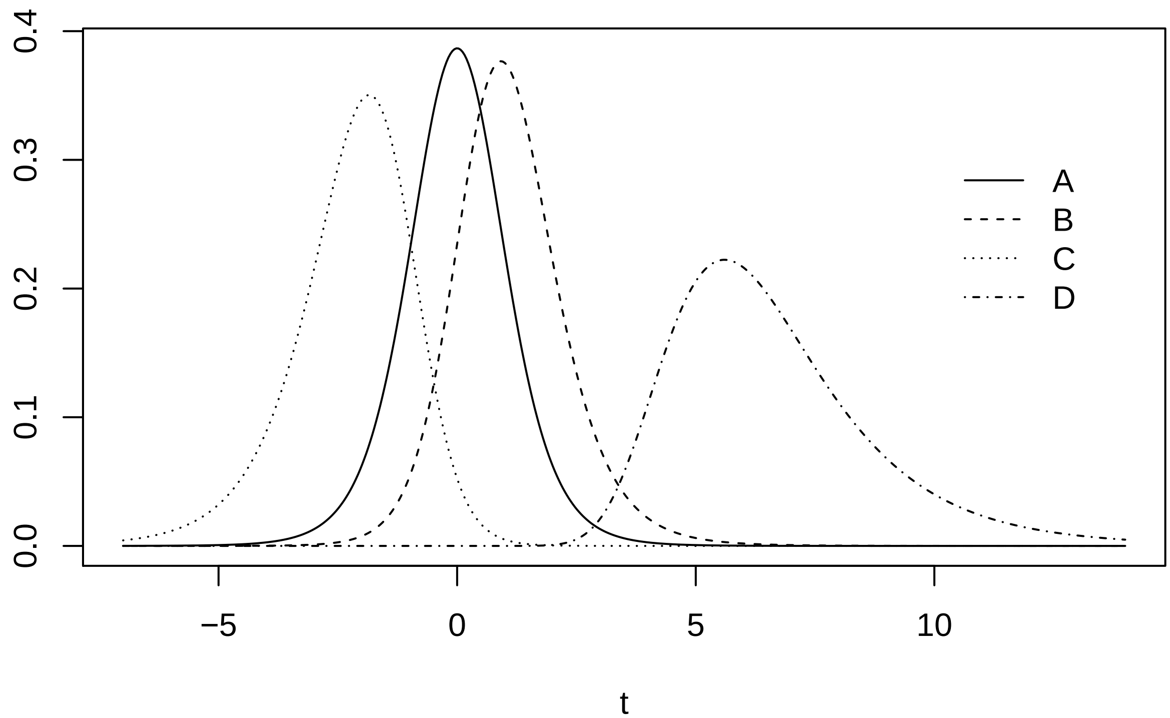
(e) Judge each of the following statements as *true* or *false*, and explain your reasoning to get credit.

i. Increasing n will guarantee that the test has a smaller maximum probability of Type I error.

ii. Increasing n will increase the power of the test for all values of μ.

iii. A larger value of σ will increase the height of the power curve to the left of μ_0.

4. The plot below shows the pdfs of four non-central t-distributions with degrees of freedom equal to 8, each with a different value of the noncentrality parameter.



The four non-centrality parameter values are $\phi_A = 0$, $\phi_B = 1$, $\phi_C = -2$, and $\phi_D = 6$. Suppose $X_1, \dots, X_9$ is a random sample from the Normal$(\mu = 1, \sigma^2 = 4)$ distribution.

(a) Identify the pdf which is the pdf of $\sqrt{9}(\bar{X}_9 - 1)/S_9$.

(b) Suppose you will reject H_0: $\mu \leq 1/3$ in favor of H_1: $\mu > 1/3$ if $\sqrt{9}(\bar{X}_9 - 1/3)/S_9 > t_{9,0.05}$. Given that the true value of the mean is $\mu = 1$, carefully shade an area of the plot such that the area is equal to the power.