STAT 513 fa 2019 Exam I

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Do not open this test until told to do so; no calculators allowed; no notes allowed; no books allowed; show your work so that partial credit may be given.

The table below gives some values of the function $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$:

z	0.841	1.282	1.645	1.96	2.326	2.576
$\Phi(z)$	0.80	0.90	0.95	0.975	.990	0.995

Result: Let $X_1, \dots, X_n$ be a random sample from the Normal(μ, σ^2) distribution. Then

$$\frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \sim t_{\phi, n-1}, \text{ where } \phi = \frac{\mu - \mu_0}{\sigma/\sqrt{n}}.$$

1. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$, where p is unknown, and suppose the hypotheses

$$H_0\colon\ p \leq 1/8 \text{ versus } H_1\colon\ p > 1/8$$

are to be tested using the test

$$\text{Reject } H_0 \text{ iff } \bar{X}_n > 1/8 + C\sqrt{(1/8)(7/8)/n}$$

for some $C > 0$, where $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$.

(a) For a given sample size n, state the distribution of $\sum_{i=1}^n X_i$.

Solution: $\sum_{i=1}^n X_i \sim \text{Binomial(n,p)}$

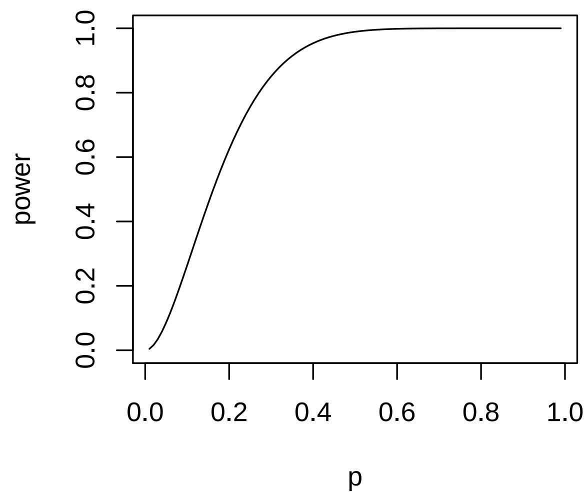
(b) Write down an expression for the power of the test as a function of p. Write the function such that it can be evaluated using the cdf of $\sum_{i=1}^n X_i$.

Solution:

$$\begin{aligned}\gamma(p) &= P_p(n^{-1}\sum_{i=1}^n X_i > 1/8 + C\sqrt{(1/8)(7/8)/n}) \\ &= 1 - P_p(Y \leq n(1/8 + C\sqrt{(1/8)(7/8)/n})), \quad Y \sim \text{Binomial}(n,p) \\ &= 1 - \sum_{x=0}^{\lfloor n(1/8 + C\sqrt{(1/8)(7/8)/n})\rfloor} \binom{n}{x} p^x (1-p)^{n-x}\end{aligned}$$

(c) Sketch the shape of the power curve.

Solution: The power curve should look like this:



(d) Give a function of $\bar{X}_n$ which, assuming $p = 1/8$, will have a sampling distribution closer and closer to the Normal$(0, 1)$ distribution for larger and larger n. *Hint: Central Limit Theorem.*

Solution: If $p = 1/8$, then the sampling distribution of

$$\frac{\sqrt{n}(\bar{X}_n - 1/8)}{\sqrt{1/8(7/8)}}$$

becomes closer and closer to the Normal$(0, 1)$ distribution as $n \to \infty$.

(e) Find the value of C such that the test given above will have size approaching 0.05 as $n \to \infty$.

Solution: The value $C = 1.645$ will give the test size approaching 0.05 as $n \to \infty$.

2. Consider a parameter $\theta \in \Theta$.

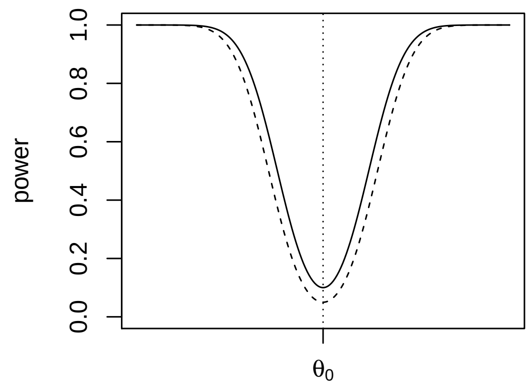
(a) Consider testing H_0: $\theta \in \Theta_0$ versus H_1: $\theta \in \Theta_1$, and suppose that instead of collecting real data related to θ, you decide you will just flip a fair coin and use the following test:

Reject H_0 iff the coin flip is "heads".

What is the size of the test?

Solution: For all $\theta \in \Theta_0$, the probability of rejecting H_0 based on this rule is $1/2$, so the size is $1/2$.

(b) Suppose you wish to test H_0: $\theta = \theta_0$ versus H_1: $\theta \neq \theta_0$, and you must choose between the two tests which have the power curves plotted below.



Discuss the relative merits of the two tests.

Solution: Well, well, it looks like we have two tests for which the power increases with the distance between θ and θ_0, which is what we would expect in a two-sided testing situation. The one corresponding to the dashed curve is a little less likely to result in Type I errors, so we might prefer it if we really don't want to falsely reject the null. On the other hand though, it has lower power than the other test over all values of θ in the alternate space, so if the null is untrue, we have a greater probability of making a Type II error. To choose between the tests we have to decide which is more important—high power or greater protection against Type I errors.

(c) Suppose you will test the hypotheses H_0: $\theta = \theta_0$ versus H_1: $\theta \neq \theta_0$ with the test

$$\text{Reject } H_0 \text{ iff } T < a \text{ or } T > b,$$

where T is some quantity you will compute on observed data and a and b are some real numbers. Suppose that when $\theta = \theta_0$, the quantity T has a distribution with cdf given by the function F.

i. Write down the probability of a Type I error in terms of the function F.

Solution: Since a Type I error is rejecting H_0 when H_0 is true, we need to give an expression for the probability of the rejection event, $T < a$ or $T > b$ under $\theta = \theta_0$. When $\theta = \theta_0$, T has the distribution with cdf F, so we have

$$P_{\theta=\theta_0}(T < a \text{ or } T > b) = F(a) + 1 - F(b).$$

ii. How could you find values a and b such that the test has size no greater than $\alpha \in (0, 1)$?

Solution: In order for the test to have size no greater than α, we need

$$F(a) + 1 - F(b) \leq \alpha.$$

If F is continuous and monotonically increasing, we can set the size exactly equal to α by choosing a to be the $\alpha/2$ quantile of F and b to be the $1 - \alpha/2$ quantile of F. If F is not continuous and monotonically increasing, then there may not exist values a and b for which the size is exactly α. In this case we can choose values of a and b which are close to the $\alpha/2$ and $1 - \alpha/2$ quantile that ensure a size no greater than α.

3. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, σ^2 known, and consider H_0: $\mu \geq \mu_0$ versus H_1: $\mu < \mu_0$. The test

$$\text{Reject } H_0 \text{ iff } \sqrt{n}(\bar{X}_n - \mu_0)/\sigma < -z_\alpha$$

has power function given by

$$\gamma(\mu) = \Phi(-z_\alpha - \sqrt{n}(\mu - \mu_0)/\sigma).$$

(a) Carefully sketch the power curve, including a vertical line positioned at μ_0 and a horizontal line positioned at α.

Solution: The power curve should be downward sloping and the power curve, the vertical line, and the horizontal line should all intersect. The range of the vertical axis should extend from 0 to 1.

(b) Suppose it is of interest to detect a deviation from the null of size σ (that is, you wish to detect whether $\mu < \mu_0 - \sigma$) with probability 0.80 while keeping the probability of a Type I error bounded by 0.01. Find the required sample size.

Solution: We wish to find the smallest n which satisfies

$$0.80 \leq \gamma(\mu_0 - \sigma) = \Phi(-z_{0.01} - \sqrt{n}((\mu_0 - \sigma) - \mu_0)/\sigma) = \Phi(-z_\alpha - \sqrt{n}).$$

From the above we have

$$z_{0.20} \leq -z_{0.01} + \sqrt{n} \iff n^2 \geq (z_{0.20} + z_{0.01})^2 = (0.841 + 2.326)^2 = (3.167)^2$$

(c) Say whether the required sample size would increase or decrease for each of the following modifications to part (b).

i. If you kept the Type I error probability bounded by 0.05 instead of 0.01.

Solution: Then the minimum required sample size would be smaller.

ii. If you wanted to detect the deviation with probability 0.90 instead of 0.80.

Solution: Then the minimum required sample size would be larger.

iii. If you wanted to detect a deviation from the null of size $\sigma/2$ instead of σ.

Solution: Then the minimum required sample size would be larger.

(d) Suppose you collect data such that $\sqrt{n}(\bar{X}_n - \mu_0)/\sigma = -1.58$. In which of the following intervals does the p-value lie?

A. $(0, .005)$
B. $[.005, .01)$
C. $[.01, .05)$
D. $[.05, .1)$
E. $[.1, .5)$
F. $[.5, 1)$

(e) Judge each of the following statements as *true* or *false*, and explain your reasoning to get credit.

i. Increasing n will guarantee that the test has a smaller maximum probability of Type I error.

Solution: False. The sample size does not have any effect on the size of the test.

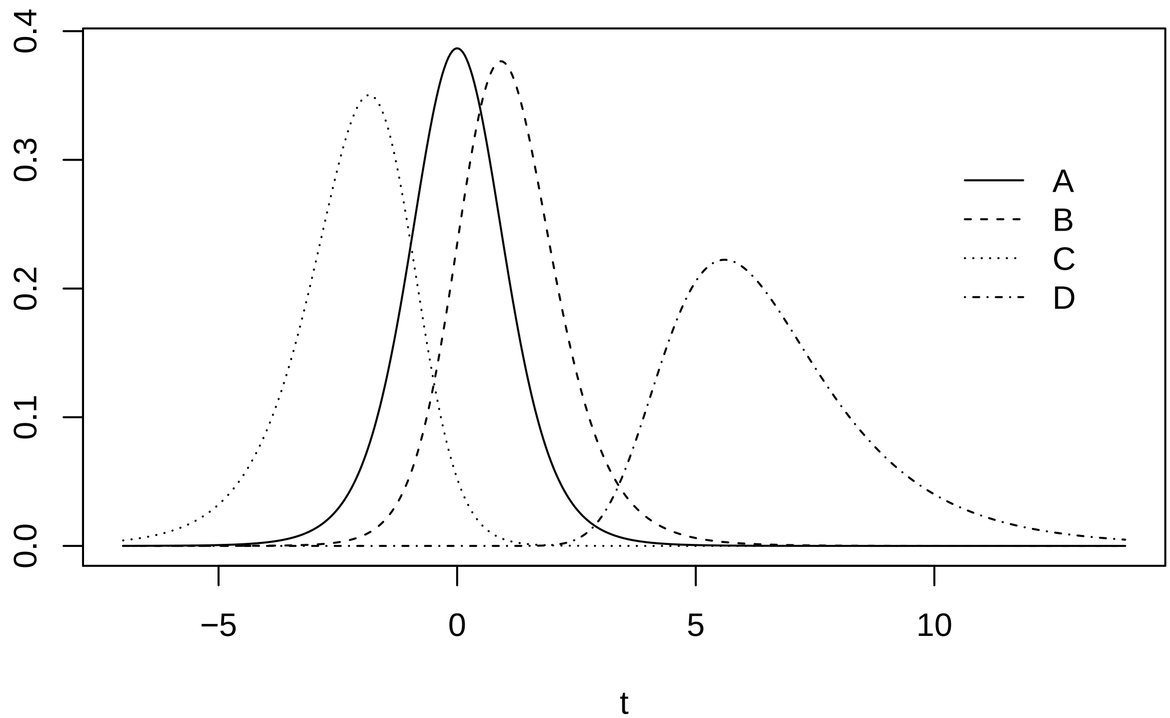
ii. Increasing n will increase the power of the test for all values of μ.

Solution: False. Increasing the sample size will increase the power of the test only for values of μ which are less than μ_0, that is only for value of μ in the alternate space.

iii. A larger value of σ will increase the height of the power curve to the left of μ_0.

Solution: False. A larger value of σ will lower the power over the alternate space, which is to the left of μ_0.

4. The plot below shows the pdfs of four non-central t-distributions with degrees of freedom equal to 8, each with a different value of the noncentrality parameter.



The four non-centrality parameter values are $\phi_A = 0$, $\phi_B = 1$, $\phi_C = -2$, and $\phi_D = 6$. Suppose $X_1, \ldots, X_9$ is a random sample from the Normal$(\mu = 1, \sigma^2 = 4)$ distribution.

(a) Identify the pdf which is the pdf of $\sqrt{9}(\bar{X}_9 - 1)/S_9$.

Solution: This is the central t distribution, with $\phi = 0$, so the answer is A.

(b) Suppose you will reject H_0: $\mu \leq 1/3$ in favor of H_1: $\mu > 1/3$ if $\sqrt{9}(\bar{X}_9 - 1/3)/S_9 > t_{9,0.05}$. Given that the true value of the mean is $\mu = 1$, carefully shade an area of the plot such that the area is equal to the power.

Solution: The quantity $\sqrt{9}(\bar{X}_9 - 1/3)/S_9$ will have the non-central t-distribution with 8 degrees of freedom and noncentrality parameter $\sqrt{9}(1 - 1/3)/2 = 1$. So the shaded region is under the dashed curve (curve B) to the right of $t_{9,0.05}$, which is found as the horizontal position such that the area under the solid curve (curve A) to the right is equal to 0.05.