STAT 513 fa 2020 Exam I

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This is a take-home test due to COVID-19. Do not communicate with classmates about the exam until after its due date/time. You may

- *Use your notes and the lecture notes.*
- *Use books.*
- *NOT work together with others.*

Write all answers on blank sheets of paper; then take pictures and merge to a PDF. Upload a single PDF to Blackboard.

1. Copy down this sentence on your answer sheet and put your signature underneath: *I have not collaborated with any other student on this exam. The work I have presented is my own.*

2. The player of a carnival game can win \$5 or \$20, and the game costs \$8 to play. Denote by p the probability of winning \$20.

   (a) For what values of p are the expected net winnings greater than 0 (your winnings minus the cost of playing the game)?

   The expected net winnings are positive if

   $$8 < p \cdot 20 + (1-p) \cdot 5 \iff p > 0.20.$$

   (b) Suppose you will play the game if you can conclude that the expected net winnings are positive. Give the null and alternate hypotheses of interest to you.

   We have

   $$H_0\colon\ p \leq 0.20 \text{ versus } H_1\colon\ p > 0.20.$$

   (c) For the following, state whether it is a correct decision or a Type I or a Type II error:

   i. You decide to play the game when $p = 0.15$.

   Type I error.

   ii. You decide not to play the game when $p = 0.30$.

   Type II error.

   iii. You decide to play the game when $p = 0.21$.

   A correct decision.

   (d) You decide to test these hypotheses by observing the outcomes for 10 other people who play the game; you will play if at least half of them win \$20.

   i. What is your test statistic?

   The number of players who win \$20.

   ii. What is your rejection region?

   $\mathcal{R} = \{5, 6, 7, 8, 9, 10\}$

   iii. Give an expression for the power function $\gamma(p)$ of your test.

Letting $Y = \#$ players who win \$20, we have $Y \sim \text{Binomial}(10, p)$, so we have

$$\gamma(p) = P(Y \geq 5) = \sum_{y=5}^{10} \binom{10}{y} p^y (1-p)^{10-y}$$

iv. What is the size of your test?

The size is

$$\gamma(0.20) = \sum_{y=5}^{10} \binom{10}{y} (0.20)^y (1-0.20)^{10-y} = \texttt{1-pbinom(4,10,0.20)} = 0.0327935.$$

v. What is the power of your test if the true probability of winning \$20 is 0.30?

The power is

$$\gamma(0.30) = \sum_{y=5}^{10} \binom{10}{y} (0.30)^y (1-0.30)^{10-y} = \texttt{1-pbinom(4,10,0.30)} = 0.1502683.$$

vi. How can you change the rejection region to reduce the size of the test?

You can change it such that more extreme data is required in order to reject the null. For example, under $\mathcal{R} = \{6, 7, 8, 9, 10\}$, the test would have a smaller size.

vii. If you reduce the size of the test, how does it affect the power when the null hypothesis is false?

If we reduce the size, the power will be smaller over the alternate space.

3. Let $Y_1, \ldots, Y_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, 1)$ and consider testing

$$H_0\colon\ \mu \leq 0 \text{ versus } H_1\colon\ \mu > 0$$

with

$$\text{Reject } H_0 \text{ iff } \sqrt{n}\bar{Y}_n > z_\alpha.$$

(a) Give the power function $\gamma(\mu)$ of the test.

We have

$$\begin{aligned}
\gamma(\mu) &= P_\mu(\sqrt{n}\bar{Y}_n > z_\alpha) \\
&= P_\mu(\sqrt{n}(\bar{Y}_n - \mu) + \sqrt{n}\mu > z_\alpha) \\
&= P(Z > z_\alpha - \sqrt{n}\mu), \quad Z \sim \text{Normal}(0,1) \\
&= 1 - \Phi(z_\alpha - \sqrt{n}\mu).
\end{aligned}$$

(b) For $\mu > 0$ what happens to the power if

i. n is decreased?

Then the power decreases.

ii. α is decreased?

Then the power decreases.

(c) For $\mu = 0$ what happens to the power if

i. n is decreased?

Then the power stays the same.

ii. α is decreased?

Then the power decreases.

(d) For $\mu < 0$ what happens to the power if

i. n is decreased?

Then the power increases.

ii. α is decreased?

Then the power decreases.

(e) If researchers wish, in the case that $\mu \geq 1$, to reject H_0 with probability at least 0.90 while keeping the Type I error probability controlled at $\alpha = 0.05$, what sample size do you recommend? Justify your answer.

The researchers can choose any n under which

$$1 - \Phi(z_{0.05} - \sqrt{n} \cdot 1) \geq 0.90 \iff \underbrace{\Phi^{-1}(1 - 0.90)}_{z_{0.90} = -z_{0.10}} \geq z_{0.05} - \sqrt{n} \iff (z_{0.10} + z_{0.05})^2 \leq n$$

so they can take any $n \geq$ `8.563847` $=$ `(qnorm(.90)+qnorm(.95))**2`. We recommend $n = 9$.

(f) For $n = 10$, what is the p-value associated with observing $\bar{Y} = 0.478$?

The p-value is

$$P(Z > \sqrt{10} \cdot 0.478) = P(Z > 1.511569) = 1 - \Phi(1.511569) = 0.06532181.$$

4. Suppose you must make phone calls to people until 3 people answer; let Y be the total number of phone calls you have to make. Let p be the probability with which each person answers the phone, and suppose you plan on testing

$$H_0\colon\ p \geq 0.10 \text{ versus } H_1\colon\ p < 0.10$$

with the test

$$\text{Reject } H_0 \text{ iff } Y > 50.$$

(a) What is the size of the test?

We have $Y \sim \text{NegativeBinomial}(r = 3, p)$. The size will be the probability of rejecting H_0 when $p = 0.10$, so we have

$$\begin{aligned}\gamma(0.10) &= P(Y > 50), \quad Y \sim \text{NegativeBinomial}(r = 3, p = 0.10)\\ &= \texttt{1 - pnbinom(50 - 3,3,0.10)}\\ &= 0.1117288.\end{aligned}$$

(b) What is the probability of a Type II error when $p = 0.08$?

A Type II error would occur if $Y \leq 50$, since then H_0 would not be rejected (but it should be). So we have

$$\begin{aligned}P(\text{Type II if } p = 0.08) &= P(Y \leq 50), \quad Y \sim \text{NegativeBinomial}(r = 3, p = 0.08)\\ &= \texttt{pnbinom(50 - 3,3,0.08)}\\ &= 0.7740257.\end{aligned}$$

5. Suppose $X_{k1}, \dots, X_{kn_k} \overset{\text{ind}}{\sim} \text{Exponential}(\lambda_k)$, $k = 1, 2$, where λ_1 and λ_2 are unknown, and let $\bar{X}_1$ and $\bar{X}_2$ be the sample means. Consider testing the hypotheses

$$H_0\colon\ \lambda_1/\lambda_2 \leq 1 \text{ vs } H_1\colon\ \lambda_1/\lambda_2 > 1$$

with the test

$$\text{Reject } H_0 \text{ iff } \bar{X}_1/\bar{X}_2 > C.$$

Note that

$$\frac{\bar{X}_1/\lambda_1}{\bar{X}_2/\lambda_2} \sim F_{2n_1,2n_2},$$

where $F_{2n_1,2n_2}$ denotes the F-distribution with numerator degrees of freedom $2n_1$ and denominator degrees of freedom $2n_2$.

(a) Suppose $\lambda_1 = 1$ and $\lambda_2 = 1$ and $n_1 = 10$ and $n_2 = 12$. Give the probability of a Type I error if the critical value $C = 2$ is used.

> If $\lambda_1 = 1$ and $\lambda_2 = 1$ and $n_1 = 10$ and $n_2 = 12$, then
>
> $$\frac{\bar{X}_1/\lambda_1}{\bar{X}_2/\lambda_2} = \frac{\bar{X}_1}{\bar{X}_2} \sim F_{20,24}.$$
>
> Then
>
> $$P(\bar{X}_1/\bar{X}_2 > 2) = 1 - F_{F_{20,24}}(2) = \texttt{1-pf(2,20,24)} = 0.05320488.$$

(b) Let $\vartheta = \lambda_1/\lambda_2$. Show that for any sample sizes n_1 and n_2 the power function $\gamma(\vartheta)$ is given by

$$\gamma(\vartheta) = 1 - F_{F_{2n_1,2n_2}}(C/\vartheta).$$

where $F_{F_{2n_1,2n_2}}$ denotes the cdf of the $F_{2n_1,2n_2}$ distribution.

> We have
>
> $$\begin{aligned}\gamma(\vartheta) &= P(\text{Rej. } H_0 \text{ if } \vartheta \text{ the true value})\\ &= P_{\lambda_1/\lambda_2=\vartheta}(\bar{X}_1/\bar{X}_2 > C)\\ &= P_{\lambda_1/\lambda_2=\vartheta}\left(\frac{\bar{X}_1/\lambda_1}{\bar{X}_2/\lambda_2}(\lambda_1/\lambda_2) > C\right)\\ &= P\left(R > (1/\vartheta)C\right), \text{ where } R \sim F_{2n_1,2n_2}\\ &= 1 - F_{F_{2n_1,2n_2}}(C/\vartheta).\end{aligned}$$
>
> distribution.

(c) Under the sample sizes $n_1 = 10$ and $n_2 = 12$, give the value C such that the size of the test will be equal to 0.01.

> The size of the test is equal to $\gamma(1)$. Setting this equal to α gives
>
> $$\alpha = \gamma(1) = 1 - F_{F_{2n_1,2n_2}}(C/1) = 1 - F_{F_{2n_1,2n_2}}(C) \iff C = F_{2n_1,2n_2,\alpha}.$$
>
> So we should set C equal to the upper α quantile of the $F_{2n_1,2n_2}$ distribution. Under the sample sizes $n_1 = 10$ and $n_2 = 12$ and with $\alpha = 0.01$, we have
>
> $$C = F_{20,24,0.01} = \texttt{qf(0.99,20,24)} = 2.737997.$$

(d) Under the sample sizes $n_1 = 10$ and $n_2 = 12$, give the power of the test under $\alpha = 0.05$ if $\lambda_1 = 1$ and $\lambda_2 = 1/2$.

The critical value of this test is $F_{20,24,0.05} =$ `qf(0.95,20,24)` $= 2.026664$. If $\lambda_1 = 1$ and $\lambda_2 = 1/2$, then $\vartheta = 2$, and the power is

$$\gamma(2) = 1 - F_{F_{20,24}}(2.026664/2) = \texttt{1 - pf(2.026664 / 2,20,24)} = 0.4825936.$$

(e) Suppose $\lambda_1 = 1$ and $\lambda_2 = 1/2$, and set $n_1 = n_2 = n$, so that the two sample sizes are equal. Find the smallest n under which the power of the test with size $\alpha = 0.05$ is at least 0.90. *Hint: You cannot solve an equation for n; you need to compute the power at different values until you find the right n.*

The power of the test with size $\alpha = 0.05$ under $n_1 = n_2 = n$ is given by

$$\gamma_n(\vartheta) = 1 - F_{F_{2n,2n}}(F_{2n,2n,0.05}/\vartheta).$$

We have $\vartheta = 2$ when $\lambda_1 = 1$ and $\lambda_2 = 1/2$. If you try many different values of n, you find that $n = 37$ is the smallest value under which the size is at least 0.90. Using the R code

$$\gamma_n(2) = \texttt{1 - pf(qf(0.95,2*n,2*n)/2,2*n,2*n)},$$

we find that $\gamma_{36}(2) = 0.8993979$ and $\gamma_{37}(2) = 0.9063827$.