# STAT 513 fa 2020 Exam II

Karl B. Gregory

*This is a take-home test due to COVID-19. Do not communicate with classmates about the exam until after its due date/time. You may*

- *Use your notes and the lecture notes.*
- *Use books.*
- *NOT work together with others.*

*Write all answers on blank sheets of paper; then take pictures and merge to a PDF. Upload a single PDF to Blackboard.*

1. Copy down this sentence on your answer sheet and put your signature underneath: *I have not collaborated with any other student on this exam. The work I have presented is my own.*

2. Let $Y_1, \dots, Y_n \overset{\text{ind}}{\sim} \text{Gamma}(\alpha, \beta)$, where $\alpha$ is <u>known</u>, and consider

$$H_0\colon\ \beta \leq \beta_0 \text{ versus } H_1\colon\ \beta > \beta_0.$$

   (a) Give the likelihood function $\mathcal{L}(\beta; Y_1, \dots, Y_n)$.

   (b) Give the log-likelihood function $\ell(\beta; Y_1, \dots, Y_n)$.

   (c) Find the maximum likelihood estimator $\hat{\beta}$ of $\beta$.

   (d) Find the restricted maximum likelihood estimator $\hat{\beta}_0$ of $\beta$ defined as

$$\hat{\beta}_0 = \underset{0<\beta\leq\beta_0}{\text{argmax}}\ \mathcal{L}(\beta; Y_1, \dots, Y_n).$$

   (e) Give the likelihood ratio $\text{LR}(Y_1, \dots, Y_n)$.

   (f) State whether the likelihood ratio is monotone increasing or decreasing in $\hat{\beta}$ when $\hat{\beta} > \beta_0$.

   (g) Use mgfs to find the distribution of $\hat{\beta}$.

   (h) Give a test that is equivalent to the likelihood ratio test of size 0.05.

   (i) Give the rejection criterion for the *asymptotic* likelihood ratio test of size 0.05 for testing

$$H_0\colon\ \beta = \beta_0 \text{ versus } H_1\colon\ \beta \neq \beta_0.$$

   (j) Let $\alpha = 2/3$, $n = 10$, $\bar{Y}_n = 1$, and $\beta_0 = 1$.

      i. Give the $p$-value of the asymptotic likelihood ratio test from part (i).

      ii. Give the $p$-value of the likelihood ratio test from part (h).

3. Let $Y_i = 10 + \varepsilon_i$ for $i = 1, \dots, 11$, where $\varepsilon_1, \dots, \varepsilon_{11} \overset{\text{ind}}{\sim} \text{Normal}(0, 1)$, and let

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}) = (-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5).$$

   Simple linear regression is to be carried out on the data pairs $(x_1, Y_1), \dots, (x_{11}, Y_{11})$, with

$$(\hat{\beta}_0, \hat{\beta}_1) = \underset{\beta_0,\beta_1}{\text{argmin}} \sum_{i=1}^{n} \left[Y_i - (\beta_0 + \beta_1 x_i)\right]^2.$$

   (a) Give the following:

      i. $\mathbb{E}\hat{\beta}_1$.

      ii. $\operatorname{Var}\hat{\beta}_1$.

      iii. $\operatorname{Cov}(\hat{\beta}_0, \hat{\beta}_1)$.

      iv. $\operatorname{Var}(\hat{\beta}_0 + 3\hat{\beta}_1)$.

      v. $P(-1/5 < \hat{\beta}_1 < 1/5)$.

(b) Suppose we observe $\sum_{i=1}^{11} \hat{\varepsilon}_i^2 = 14.29$, $\bar{Y}_n = 10.32$, $S_{YY} = 14.47$, and $r_{xY} = 0.111$. Based on this data, give

i. The value of $\hat{\beta}_1$.

ii. The value of $\hat{\beta}_0$.

iii. A 95% confidence interval for $\beta_1$.

iv. A 95% confidence interval for the height of the regression function at $x_{\text{new}} = 1/2$.

v. A 95% prediction interval for a new value $Y_{\text{new}}$ of the response with $x_{\text{new}} = 1/2$.

4. Suppose 12 individuals will be randomly assigned to a placebo group or to one of two treatment groups, such that 4 individuals will receive the placebo, 4 will receive treatment one, and 4 will receive treatment two. Clinical response values $Y_1, \ldots, Y_{12}$ will be recorded and the multiple linear regression model

$$Y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \varepsilon_i, \quad i = 1, \ldots, 12,$$

where $\varepsilon_1, \ldots, \varepsilon_n \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$, and

$$(x_{1i}, x_{2i}, x_{31}) = \begin{cases} (1,0,0), & \text{if individual } i \text{ in placebo group} \\ (0,1,0), & \text{if individual } i \text{ in treatment one group} \\ (0,0,1), & \text{if individual } i \text{ in treatment two group,} \end{cases}$$

will be fit to the data (note that the model does not have an intercept). Let $\bar{Y}_0$, $\bar{Y}_1$ and $\bar{Y}_2$ be the mean responses observed in the placebo, treatment one, and treatment two groups, respectively, and suppose the design matrix $\mathbf{X}$ of the study is given by

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Give $(\mathbf{X}^T\mathbf{X})^{-1}$.

(b) Give $\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$.

(c) Consider the hypotheses

$$H_0\colon\ \beta_2 = \beta_3 \text{ versus } H_1\colon\ \beta_2 \neq \beta_3.$$

i. Give an interpretation of the null hypothesis $H_0$ in words in the context of the problem.

ii. What is the degrees of freedom of the chi-squared distribution from which we would find the critical value for the asymptotic likelihood ratio test?