STAT 513 hw 1

1. You are interested in the probability of the outcome ⚅ when rolling a six-sided die. Answer the following questions using p to denote the probability of this outcome.
   (a) You suspect that for this die the outcome ⚅ does not occur with the same probability as with a fair die. Formulate the null and alternative hypotheses based on which you may make a statistical inference regarding your suspicion.
   (b) You suspect that for this die the outcome ⚅ occurs with greater probability than with a fair die. Formulate the null and alternative hypotheses based on which you may make a statistical inference regarding your suspicion.
   (c) You suspect that for this die the outcome ⚅ occurs with lesser probability than with a fair die. Formulate the null and alternative hypotheses based on which you may make a statistical inference regarding your suspicion.
   (d) You believe that for this die the outcome ⚅ occurs with the same probability as with a fair die.
      i. How might you collect evidence in support of your belief?
      ii. Which is easier: to collect evidence in favor of balancedness or to collect evidence against balancedness?

2. You wish to test whether a coin lands "heads" and "tails" with the same probability. Use p to denote the probability that the coin lands "heads".
   (a) State the relevant null and alternate hypotheses.
   (b) Suppose you choose to reject the null hypothesis when in 10 tosses you get more than 7 or less than 3 "heads".
      i. What is the power of your test when the true probability of "heads" is 0.6?
      ii. What is the probability that your test results in a Type II error when the true probability of "heads" is 0.3?
      iii. Write an expression for the power function $\gamma(p)$ of the test.
      iv. What is the size of the test?
      v. Make a plot of the power $\gamma(p)$ against p for $p = 0.01, 0.02, \ldots, 0.99$ (Use R).
      vi. At what value of p is the power equal to the size?
   (c) Propose a test based on 20 tosses which has size less than or equal to 0.05.
   (d) Plot the power curve of your test.

3. Suppose $X_1, \ldots, X_n$ is a random sample from the Normal$(\mu, 4)$ distribution and consider three situations:
   1. Test H_0: $\mu \leq 0$ versus H_1: $\mu > 0$ with test: Reject H_0 iff $\sqrt{n}\bar{X}_n/2 > C_1$
   2. Test H_0: $\mu = 0$ versus H_1: $\mu \neq 0$ with test: Reject H_0 iff $|\sqrt{n}\bar{X}_n/2| > C_2$
   3. Test H_0: $\mu \geq 0$ versus H_1: $\mu < 0$ with test: Reject H_0 iff $\sqrt{n}\bar{X}_n/2 < C_3$

   (a) Find an expression for the power functions $\gamma_1(\mu)$, $\gamma_2(\mu)$, and $\gamma_3(\mu)$ for the three tests. Use the notation $\Phi(z) = P(Z < z)$, where $Z \sim$ Normal$(0, 1)$.

(b) Find the values C_1, C_2, and C_3 such that each of the above tests has size equal to 0.05.

(c) For $n = 10$ and the choices of C_1,C_2, and C_3 from part (b), plot the power functions $\gamma_1(\mu)$, $\gamma_2(\mu)$, and $\gamma_3(\mu)$ against μ on the same plot for μ between -3 and 3 (Use R).

(d) Suppose you are a researcher interested in showing that $\mu > 0$. Is it better to use test 1. or test 2.? Explain your answer.

4. Let $X_1, \ldots, X_n$ be a random sample from the Uniform$(0, \theta)$ distribution, where $\theta \in (0, \infty)$ is unknown. The values $X_1, \ldots, X_n$ could be time intervals between occurrences of some phenomenon, so that θ would represent the maximum possible duration of any interval. A researcher wishes to show that the maximum possible duration is less than 1, that is to test the hypotheses H_0: $\theta \geq 1$ versus H_1: $\theta < 1$.

(a) Consider the test

$$\text{Reject } H_0 \text{ iff } X_{(n)} < 2/3,$$

where $X_{(n)}$ is the largest order statistic.

i. For a sample size of 10, what is the probability of rejecting H_0 when $\theta = 5/4$?

ii. For a sample size of 10, what is the probability of committing a Type II error when $\theta = 3/4$?

iii. What is the power of the test when $\theta \in (0, 2/3]$?

iv. For a sample size of 10, give an expression for the power function of the test.

v. For a sample size of 10, give the size of the test.

(b) Consider the test

$$\text{Reject } H_0 \text{ iff } X_{(n)} < C$$

for $0 < C < 1$.

i. Find an expression for the power function $\gamma(\theta)$ of the test (involves n and θ).

ii. Give an expression for the size of the test in terms of n and C.

iii. Give an expression for the value C which will make the test have size equal to α for any $\alpha \in (0, 1)$.

iv. For a sample size of $n = 20$, find the value C such that the test has size equal to 0.01.