STAT 513 hw 1

1. You are interested in the probability of the outcome ⚅ when rolling a six-sided die. Answer the following questions using p to denote the probability of this outcome.

   (a) You suspect that for this die the outcome ⚅ does not occur with the same probability as with a fair die. Formulate the null and alternative hypotheses based on which you may make a statistical inference regarding your suspicion.

   > H_0: $p = 1/6$ versus H_1: $p \neq 1/6$

   (b) You suspect that for this die the outcome ⚅ occurs with greater probability than with a fair die. Formulate the null and alternative hypotheses based on which you may make a statistical inference regarding your suspicion.

   > H_0: $p \leq 1/6$ versus H_1: $p > 1/6$

   (c) You suspect that for this die the outcome ⚅ occurs with lesser probability than with a fair die. Formulate the null and alternative hypotheses based on which you may make a statistical inference regarding your suspicion.

   > H_0: $p \geq 1/6$ versus H_1: $p < 1/6$

   (d) You believe that for this die the outcome ⚅ occurs with the same probability as with a fair die.

      i. How might you collect evidence in support of your belief?

      > Flip the coin very many times; if "heads" turns up close to half the time, it seems to support the claim; however, someone could always say, "well, maybe the probability of heads is 0.5001."

      ii. Which is easier: to collect evidence in favor of balancedness or to collect evidence against balancedness?

      > Even if from a very large number of tosses "heads" comes up close to half the time, one can never defend a claim that the coin is perfectly balanced: an objector can always posit that the true probability of "heads" is 0.50001 or 0.5000001, and then one has to do more tosses *ad infinitum*. Thus it is easier to collect evidence against the claim of balancedness than in favor of it.

2. You wish to test whether a coin lands "heads" and "tails" with the same probability. Use p to denote the probability that the coin lands "heads".

   (a) State the relevant null and alternate hypotheses.

H_0: $p = 1/2$ versus H_0: $p \neq 1/2$.

(b) Suppose you choose to reject the null hypothesis when in 10 tosses you get more than 7 or less than 3 "heads".

i. What is the power of your test when the true probability of "heads" is 0.6?

`pbinom(2,10,.6) + 1 - pbinom(7,10,.6)` $= 0.1795843$

ii. What is the probability that your test results in a Type II error when the true probability of "heads" is 0.3?

`1-(pbinom(2,10,.3) + 1 - pbinom(7,10,.3))` $= 0.6156268$

iii. Write an expression for the power function $\gamma(p)$ of the test.

$\gamma(p) = \sum_{y=0}^{2} \binom{10}{y} p^y (1-p)^{10-y} + \sum_{y=8}^{10} \binom{10}{y} p^y (1-p)^{10-y}$

iv. What is the size of the test?

$$\gamma(1/2) = \sum_{y=0}^{2} \binom{10}{y} (1/2)^y (1-1/2)^{10-y} + \sum_{y=8}^{10} \binom{10}{y} (1/2)^y (1-1/2)^{10-y}$$

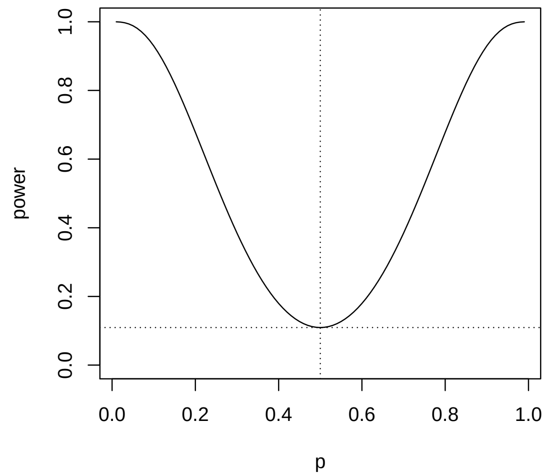
$= $ `sum(dbinom(0:2,10,.5),dbinom(8:10,10,.5))`

$= 0.109375.$

v. Make a plot of the power $\gamma(p)$ against p for $p = 0.01, 0.02, \ldots, 0.99$ (Use R).

```
p.seq <- seq(.01,.99,length=99)
power <- pbinom(2,10,p.seq) + 1 - pbinom(7,10,p.seq)

plot(p.seq,power,type="l",ylim=c(0,1),xlab="p")
abline(v=0.5,lty=3)                # vert line at null value
abline(h=0.109375,lty=3)# horiz line at size
```



vi. At what value of p is the power equal to the size?

The power is equal to the size at $p = 1/2$, which is the value of p specified in the null hypothesis.

(c) Propose a test based on 20 tosses which has size less than or equal to 0.05.

Consider tests of the form

$$\text{Reject } H_0 \text{ iff } X_1 + \cdots + X_{20} \in \mathcal{R}$$

for different rejection regions $\mathcal{R}$. One option is to choose

$$\mathcal{R} = \{0, 1, 2, 3, 4, 5\} \cup \{15, 16, 17, 18, 19, 20\},$$

with which the test has size

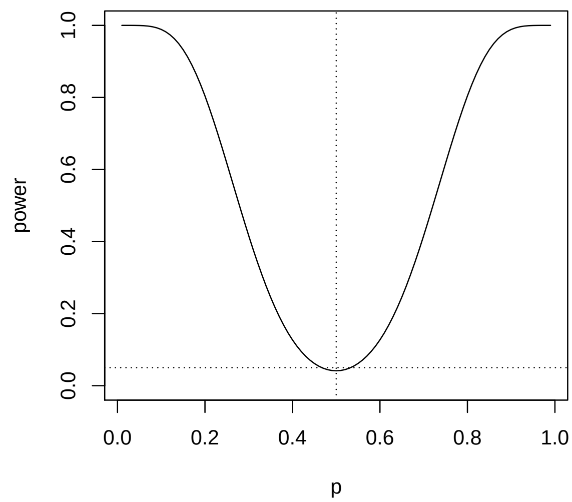
`pbinom(5,20,.5) + 1 - pbinom(14,20,.5)` $= 0.04138947 < 0.05$

(d) Plot the power curve of your test.

```
p.seq <- seq(.01,.99,length=99)
power <- pbinom(5,20,p.seq) + 1 - pbinom(14,20,p.seq)

plot(p.seq,power,type="l",ylim=c(0,1),xlab="p")
```

```
abline(v=0.5,lty=3)  # vert line at null value
abline(h=0.05,lty=3) # horiz line at max. allowed size
```



3. Suppose $X_1, \ldots, X_n$ is a random sample from the Normal$(\mu, 4)$ distribution and consider three situations:

   1. Test H_0: $\mu \leq 0$ versus H_1: $\mu > 0$ with test: Reject H_0 iff $\sqrt{n}\bar{X}_n/2 > C_1$
   2. Test H_0: $\mu = 0$ versus H_1: $\mu \neq 0$ with test: Reject H_0 iff $|\sqrt{n}\bar{X}_n/2| > C_2$
   3. Test H_0: $\mu \geq 0$ versus H_1: $\mu < 0$ with test: Reject H_0 iff $\sqrt{n}\bar{X}_n/2 < C_3$

   (a) Find an expression for the power functions $\gamma_1(\mu)$, $\gamma_2(\mu)$, and $\gamma_3(\mu)$ for the three tests. Use the notation $\Phi(z) = P(Z < z)$, where $Z \sim$ Normal$(0, 1)$.

   1. The power for 1. is given by

$$\begin{aligned}
\gamma_1(\mu) &= P_\mu(\bar{X}_n/(2/\sqrt{n}) > C_1) \\
&= P_\mu((\bar{X}_n - \mu)/(2/\sqrt{n}) > C_1 - \mu/(2/\sqrt{n})) \\
&= P(Z > C_1 - \mu/(2/\sqrt{n})), \quad Z \sim \text{Normal}(0, 1) \\
&= 1 - \Phi(C_1 - \mu/(2/\sqrt{n})).
\end{aligned}$$

2. The power for 2. is given by

$$\begin{aligned}\gamma_2(\mu) &= P_\mu(|\bar{X}_n/(2/\sqrt{n})| > C_2)\\ &= 1 - P_\mu(-C_2 < \bar{X}_n/(2/\sqrt{n}) < C_2)\\ &= 1 - P_\mu(-C_2 - \mu/(2/\sqrt{n}) < (\bar{X}_n - \mu)/(2/\sqrt{n}) < C_2 - \mu/(2/\sqrt{n}))\\ &= 1 - P(-C_2 - \mu/(2/\sqrt{n}) < Z < C_2 - \mu/(2/\sqrt{n})), \quad Z \sim \text{Normal}(0,1)\\ &= 1 - [P(Z < C_2 - \mu/(2/\sqrt{n})) - P(Z < -C_2 - \mu/(2/\sqrt{n}))]\\ &= 1 - [\Phi(C_2 - \mu/(2/\sqrt{n})) - \Phi(-C_2 - \mu/(2/\sqrt{n}))].\end{aligned}$$

3. The power for 3. is given by

$$\begin{aligned}\gamma_3(\mu) &= P_\mu(\bar{X}_n/(2/\sqrt{n}) < C_3)\\ &= P_\mu((\bar{X}_n - \mu)/(2/\sqrt{n}) > C_3 - \mu/(2/\sqrt{n}))\\ &= P(Z < C_3 - \mu/(2/\sqrt{n})), \quad Z \sim \text{Normal}(0,1)\\ &= \Phi(C_3 - \mu/(2/\sqrt{n})).\end{aligned}$$

(b) Find the values C_1, C_2, and C_3 such that each of the above tests has size equal to 0.05.

1. From part (a), the power is given by

$$\gamma_1(\mu) = P(Z > C_1 - \mu/(2/\sqrt{n})), \quad Z \sim \text{Normal}(0,1),$$

so the size is

$$\sup_{\mu \le 0} \gamma_1(\mu) = \gamma_1(0) = P(Z > C_1)$$

Setting the size equal to 0.05 gives

$$0.05 = P(Z > C_1) \iff C_1 = 1.644854 = \texttt{qnorm(.95)}.$$

2. From part (a), the power is given by

$$\gamma(\mu) = 1 - [P(Z < C_2 - \mu/(2/\sqrt{n})) - P(Z < -C_2 - \mu/(2/\sqrt{n}))],$$

so the size is

$$\sup_{\mu \in \{0\}} \gamma_2(\mu) = \gamma_2(0) = 1 - [P(Z < C_2) - P(Z < -C_2)] = 2[1 - P(Z < C_2)]$$

Setting the size equal to 0.05 gives

$$0.05 = 2[1 - P(Z < C_2)] \iff P(Z < C_2) = .975 \iff C_2 = 1.959964 = \texttt{qnorm(.975)}$$

3. From part (a), the power is given by

$$\gamma_3(\mu) = P(Z < C_3 - \mu/(2/\sqrt{n})), \quad Z \sim \text{Normal}(0,1),$$

so the size is

$$\sup_{\mu \geq 0} \gamma_3(\mu) = \gamma_3(0) = P(Z < C_3)$$

Setting the size equal to 0.05 gives

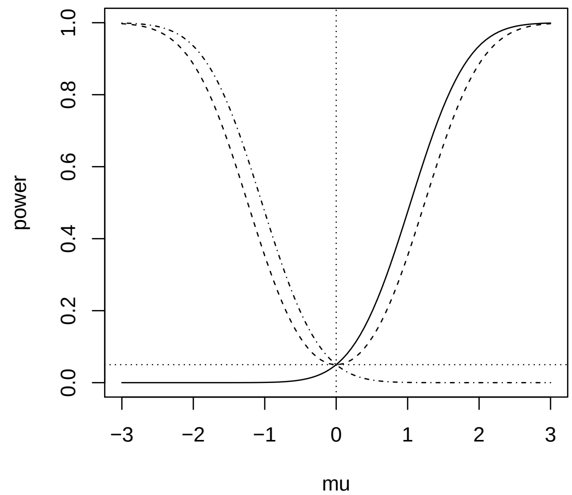
$$0.05 = P(Z < C_3) \iff C_3 = -1.644854 = \texttt{qnorm(.05)}.$$

(c) For $n = 10$ and the choices of C_1,C_2, and C_3 from part (b), plot the power functions $\gamma_1(\mu)$, $\gamma_2(\mu)$, and $\gamma_3(\mu)$ against μ on the same plot for μ between -3 and 3 (Use R).

```
mu.seq <- seq(-3,3,length=100)

power1 <- 1-pnorm(1.644854-mu.seq/(2/sqrt(10)))
power2 <- 1-(pnorm(1.959964-mu.seq/(2/sqrt(10)))
                    -pnorm(-1.959964-mu.seq/(2/sqrt(10))))
power3 <- pnorm(-1.644854-mu.seq/(2/sqrt(10)))

plot(mu.seq,power1,type="l",ylim=c(0,1),xlab="mu",ylab="power")
lines(mu.seq,power2,ylim=c(0,1),lty=2)
lines(mu.seq,power3,ylim=c(0,1),lty=4)
abline(v=0,lty=3)    # vert line at null value
abline(h=0.05,lty=3) # horiz line at size
```



(d) Suppose you are a researcher interested in showing that $\mu > 0$. Is it better to use test 1. or test 2.? Explain your answer.

Test 1 is better if one is interested in claiming that $\mu > 0$ because test 1 has more power than test 2 when $\mu > 0$.

4. Let $X_1, \dots, X_n$ be a random sample from the Uniform$(0, \theta)$ distribution, where $\theta \in (0, \infty)$ is unknown. The values $X_1, \dots, X_n$ could be time intervals between occurrences of some phenomenon, so that θ would represent the maximum possible duration of any interval. A researcher wishes to show that the maximum possible duration is less than 1, that is to test the hypotheses H_0: $\theta \geq 1$ versus H_1: $\theta < 1$.

(a) Consider the test

$$\text{Reject } H_0 \text{ iff } X_{(n)} < 2/3,$$

where $X_{(n)}$ is the largest order statistic.

i. For a sample size of 10, what is the probability of rejecting H_0 when $\theta = 5/4$?

The density of the 10th order statistic is

$$X_{(10)} \sim f_{X_{(10)}}(x) = 10x^9/\theta^{10}\mathbb{1}(0 < x < \theta),$$

so that

$$P_{\theta=5/4}(X_{(10)} < 2/3) = \int_{-\infty}^{2/3} 10x^9/(5/4)^{10}\mathbb{1}(0 < x < 5/4)dx = (2/3)^{10}(4/5)^{10} = 0.001862.$$

ii. For a sample size of 10, what is the probability of committing a Type II error when $\theta = 3/4$?

A Type II error is failing to reject H_0 when it is false; it is false if $\theta = 3/4$. So the probability is one minus the power; that is

$$
\begin{aligned}
1 - P_{\theta=3/4}(X_{(10)} < 2/3) &= 1 - \int_{-\infty}^{2/3} 10x^9/(3/4)^{10}\mathbb{1}(0 < x < 3/4)dx \\
&= 1 - \int_{0}^{2/3} 10x^9/(3/4)^{10}dx \\
&= 1 - (2/3)^{10}(4/3)^{10} \\
&= 1 - 0.3079461 \\
&= 0.6920539.
\end{aligned}
$$

iii. What is the power of the test when $\theta \in (0, 2/3]$?

If $\theta \in (0, 2/3]$ then the largest order statistic *must* be less than $2/3$, so the power is equal to 1.

iv. For a sample size of 10, give an expression for the power function of the test.

$$
\begin{aligned}
\gamma(\theta) &= P_\theta(X_{(10)} < 2/3) \\
&= \int_{-\infty}^{2/3} 10x^9/\theta^{10}\mathbb{1}(0 < x < \theta)dx \\
&= \begin{cases} 1 & 0 < \theta < 2/3 \\ (2/3)^{10}(1/\theta)^{10} & 2/3 \le \theta < \infty \end{cases}
\end{aligned}
$$

v. For a sample size of 10, give the size of the test.

$$\sup_{\theta \ge 1} \gamma(\theta) = \gamma(1) = (2/3)^{10} = 0.01734153.$$

(b) Consider the test

$$\text{Reject } H_0 \text{ iff } X_{(n)} < C$$

for $0 < C < 1$.

i. Find an expression for the power function $\gamma(\theta)$ of the test (involves n and θ).

$$\begin{aligned}\gamma(\theta) &= P_\theta(X_{(n)} < C) \\ &= \int_{-\infty}^{C} nx^{n-1}/\theta^n \mathbb{1}(0 < x < \theta)dx \\ &= \begin{cases} 1 & 0 < \theta < C \\ (C/\theta)^n & C \leq \theta < \infty \end{cases}\end{aligned}$$

ii. Give an expression for the size of the test in terms of n and C.

$$\sup_{\theta \geq 1} \gamma(\theta) = \gamma(1) = C^n$$

iii. Give an expression for the value C which will make the test have size equal to α for any $\alpha \in (0, 1)$.

We have $\alpha = C^n \iff C = \alpha^{1/n}$.

iv. For a sample size of $n = 20$, find the value C such that the test has size equal to 0.01.

$C = (0.01)^{(1/20)} = 0.7943282$.