STAT 513 hw 4

1. Let $X_1, \dots, X_n$ be a random sample from the $\text{Normal}(\mu, \sigma^2)$ distribution, where μ and σ^2 are unknown, and suppose we are interested in testing H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$.

   (a) Give the likelihood function $L(\mu, \sigma^2; X_1, \dots, X_n)$.

   (b) Give the log-likelihood function $\ell(\mu, \sigma^2; X_1, \dots, X_n)$.

   (c) Find the maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}^2$ of μ and σ^2.

   (d) Find an expression for $\hat{\sigma}_0^2$, where

   $$\hat{\sigma}_0^2 = \text{argmax}_{\sigma^2} L(\mu_0, \sigma^2; X_1, \dots, X_n).$$

   (e) Give the likelihood ratio.

   (f) Show that the likelihood ratio simplifies to

   $$\text{LR}(X_1, \dots, X_n) = \left[\frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{\sum_{i=1}^n (X_i - \mu_0)^2} \right]^{n/2}$$

   (g) Show that for any $c \in [0, 1]$ the rejection criterion of the likelihood ratio test, $\text{LR}(X_1, \dots, X_n) < c$, is equivalent to

   $$\sqrt{n}|\bar{X}_n - \mu_0|/S_n > c^*$$

   for some c^*.

   (h) Use the fact that

   $$X_1, \dots, X_n \overset{\text{iid}}{\sim} \text{Normal}(\mu, \sigma^2) \implies \sqrt{n}(\bar{X}_n - \mu)/S_n \sim t_{n-1}$$

   to find a value c^* such that the likelihood ratio test has size equal to α for any $\alpha \in (0, 1)$.

2. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Exponential}(\lambda)$ and consider

   $$H_0\colon \lambda \leq \lambda_0 \text{ versus } H_1\colon \lambda > \lambda_0.$$

   (a) Show that the likelihood function $\mathcal{L}(\lambda; X_1, \dots, X_n)$ for λ can be written as

   $$\mathcal{L}(\lambda; X_1, \dots, X_n) = \lambda^{-n} \exp(-n\bar{X}_n/\lambda).$$

   (b) Give the log-likelihood function $\ell(\lambda; X_1, \dots, X_n)$ for λ.

   (c) Find the MLE $\hat{\lambda}$ of λ.

   (d) For $n = 10$, and $\bar{X}_n = 5$, make a plot in R of the log-likelihood $\ell(\lambda; X_1, \dots, X_n)$ over $\lambda \in (5/2, 10)$. Add a vertical line at the position of $\bar{X}_n$.

(e) Find the restricted MLE $\hat{\lambda}_0$ under H_0. That is, find

$$\hat{\lambda}_0 = \underset{\lambda \le \lambda_0}{\operatorname{argmax}}\ \mathcal{L}(\lambda; X_1, \ldots, X_n).$$

Hint: It must be piecewise defined for the cases $\bar{X}_n \le \lambda_0$ *and* $\bar{X}_n > \lambda_0$.

(f) Give the likelihood ratio.

(g) Show that the likelihood ratio test is equivalent to the test

$$\text{Reject } H_0 \text{ iff } \bar{X}_n > c^*$$

for some value $c^* > 0$. *Hint: The function* ze^{-z} *is stricty decreasing in* z *for all* $z > 1$.

(h) Use that fact that $\bar{X}_n \sim \text{Gamma}(n, \lambda/n)$ to find the critical value c^* in part (g) such that the test has size α.

(i) Under $n = 20$ and $\lambda_0 = 5$,

  i. give the critical value c^* in part (g) such that the test has size $\alpha = 0.05$.

  ii. give the p-value corresponding to the observation $\bar{X}_{20} = 5.7$.