STAT 513 hw 4

1. Let $X_1, \dots, X_n$ be a random sample from the $\text{Normal}(\mu, \sigma^2)$ distribution, where μ and σ^2 are unknown, and suppose we are interested in testing H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$.

   (a) Give the likelihood function $L(\mu, \sigma^2; X_1, \dots, X_n)$.

   $$\begin{aligned} L(\mu, \sigma^2; X_1, \dots, X_n) &= \textstyle\prod_{i=1}^n (2\pi)^{-1/2} \sigma^{-1} \exp[-(1/2)(X_i - \mu)^2/\sigma^2] \\ &= (2\pi)^{-n/2} \sigma^{-n} \exp[-(1/2) \textstyle\sum_{i=1}^n (X_i - \mu)^2/\sigma^2]. \end{aligned}$$

   (b) Give the log-likelihood function $\ell(\mu, \sigma^2; X_1, \dots, X_n)$.

   $$\ell(\mu, \sigma^2; X_1, \dots, X_n) = -(n/2)\log(2\pi) - (n/2)\log \sigma^2 - (1/2) \textstyle\sum_{i=1}^n (X_i - \mu)^2/\sigma^2,$$

   (c) Find the maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}^2$ of μ and σ^2.

   $$\hat{\mu} = \bar{X}_n \text{ and } \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2$$

   (d) Find an expression for $\hat{\sigma}_0^2$, where

   $$\hat{\sigma}_0^2 = \text{argmax}_{\sigma^2} L(\mu_0, \sigma^2; X_1, \dots, X_n).$$

   $$\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \mu_0)^2$$

   (e) Give the likelihood ratio.

   $$\frac{L(\mu_0, \hat{\sigma}_0^2; X_1, \dots, X_n)}{L(\hat{\mu}, \hat{\sigma}^2; X_1, \dots, X_n)} = \frac{(2\pi)^{-n/2} e^{-n/2} [n^{-1} \sum_{i=1}^n (X_i - \mu_0)^2]^{-n/2}}{(2\pi)^{-n/2} e^{-n/2} [n^{-1} \sum_{i=1}^n (X_i - \hat{\mu})^2]^{-n/2}}$$

   (f) Show that the likelihood ratio simplifies to

   $$\text{LR}(X_1, \dots, X_n) = \left[\frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{\sum_{i=1}^n (X_i - \mu_0)^2}\right]^{n/2}$$

(g) Show that for any $c \in [0,1]$ the rejection criterion of the likelihood ratio test, $\text{LR}(X_1,\dots,X_n) < c$, is equivalent to

$$\sqrt{n}|\bar{X}_n - \mu_0|/S_n > c^*$$

for some c^*.

$$\begin{aligned}
\left[\frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{\sum_{i=1}^n (X_i - \mu_0)^2}\right]^{n/2} &< c \\
\iff \frac{\sum_{i=1}^n (X_i - \mu_0)^2}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} &> (1/c)^{2/n} \\
\iff \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2 + n(\bar{X}_n - \mu_0)^2}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} &> (1/c)^{2/n} \\
\iff 1 + \frac{n(\bar{X}_n - \mu_0)^2}{(n-1)S_n^2} &> (1/c)^{2/n} \\
\iff \sqrt{n}|\bar{X}_n - \mu_0|/S_n &> \underbrace{\sqrt{(n-1)[(1/c)^{2/n} - 1]}}_{c^*}.
\end{aligned}$$

(h) Use the fact that

$$X_1,\dots,X_n \overset{\text{iid}}{\sim} \text{Normal}(\mu,\sigma^2) \implies \sqrt{n}(\bar{X}_n - \mu)/S_n \sim t_{n-1}$$

to find a value c^* such that the likelihood ratio test has size equal to α for any $\alpha \in (0,1)$.

Choose $c^* = t_{n-1,\alpha/2}$.

2. Let $X_1,\dots,X_n \overset{\text{ind}}{\sim} \text{Exponential}(\lambda)$ and consider

$$H_0\colon \lambda \leq \lambda_0 \text{ versus } H_1\colon \lambda > \lambda_0.$$

(a) Show that the likelihood function $\mathcal{L}(\lambda; X_1,\dots,X_n)$ for λ can be written as

$$\mathcal{L}(\lambda; X_1,\dots,X_n) = \lambda^{-n}\exp(-n\bar{X}_n/\lambda).$$

We have

$$\mathcal{L}(\lambda; X_1,\dots,X_n) = \prod_{i=1}^n \frac{1}{\lambda}\exp(-X_i/\lambda) = \lambda^{-n}\exp\left[\sum_{i=1}^n X_i/\lambda\right]$$

(b) Give the log-likelihood function $\ell(\lambda; X_1, \ldots, X_n)$ for λ.

We have

$$\ell(\lambda; X_1, \ldots, X_n) = -n \log \lambda - n\bar{X}_n/\lambda.$$

(c) Find the MLE $\hat{\lambda}$ of λ.

We have

$$\hat{\lambda} = \bar{X}_n,$$

since

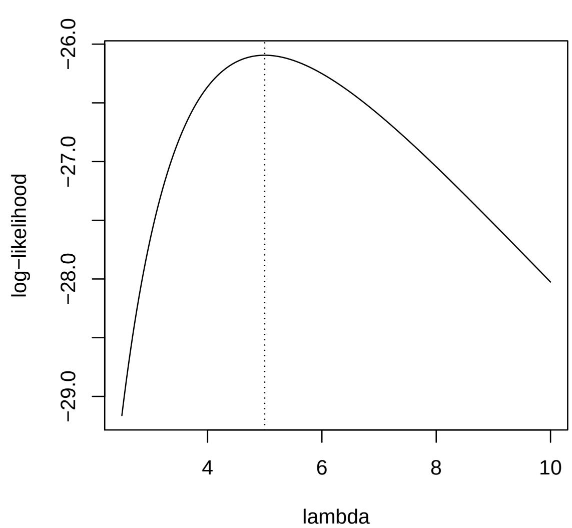
$$\frac{\partial}{\partial \lambda}\ell(\lambda; X_1, \ldots, X_n) = -\frac{n}{\lambda} + \frac{n\bar{X}_n}{\lambda^2} = 0 \iff \lambda = \bar{X}_n.$$

(d) For $n = 10$, and $\bar{X}_n = 5$, make a plot in R of the log-likelihood $\ell(\lambda; X_1, \ldots, X_n)$ over $\lambda \in (5/2, 10)$. Add a vertical line at the position of $\bar{X}_n$.

The R code

```
n <- 10
Xbar <- 5
lambda <- seq(5/2,2*5,length = 200)
LL <- - n * log(lambda) - n * Xbar / lambda
plot(LL ~ lambda,type = "l")
abline(v = Xbar, lty =3)
```

produces the plot



(e) Find the restricted MLE $\hat{\lambda}_0$ under H_0. That is, find

$$\hat{\lambda}_0 = \underset{\lambda \le \lambda_0}{\text{argmax}} \, \mathcal{L}(\lambda; X_1, \ldots, X_n).$$

Hint: It must be piecewise defined for the cases $\bar{X}_n \le \lambda_0$ *and* $\bar{X}_n > \lambda_0$.

If $\bar{X}_n \le \lambda_0$, that is if $\bar{X}_n$ is found in the null space, then $\hat{\lambda}_0 = \bar{X}_n$, since $\bar{X}_n$ maximizes the likelihood; however, if $\bar{X}_n > \lambda$, we have $\hat{\lambda}_0 = \lambda_0$, since the likelihood is strictly increasing (slope of log-likelihood is positive) for all $\lambda < \bar{X}_n$. So we write

$$\hat{\lambda}_0 = \begin{cases} \lambda_0, & \bar{X}_n > \lambda_0 \\ \bar{X}_n, & \bar{X}_n \le \lambda_0. \end{cases}$$

(f) Give the likelihood ratio.

If the MLE $\bar{X}_n$ of λ is in the null space $(0, \lambda_0]$, then the LRT is equal to 1. So we have

$$\text{LR}(X_1, \ldots, X_n) = \frac{\hat{\lambda}_0^{-n} \exp(-n\bar{X}_n/\hat{\lambda}_0)}{\hat{\lambda}^{-n} \exp(-n\bar{X}_n/\hat{\lambda})} = \begin{cases} \left[\dfrac{\bar{X}_n}{\lambda_0} \exp\left(-\dfrac{\bar{X}_n}{\lambda_0}\right)\right]^n e^n, & \bar{X}_n > \lambda_0 \\ 1, & \bar{X}_n \le \lambda_0. \end{cases}$$

(g) Show that the likelihood ratio test is equivalent to the test

$$\text{Reject } H_0 \text{ iff } \bar{X}_n > c^*$$

for some value $c^* > 0$. *Hint: The function* ze^{-z} *is stricty decreasing in* z *for all* $z > 1$.

For $\bar{X}_n > \lambda_0$, the LRT is monotone decreasing in $\bar{X}_n$ (a strictly decreasing function). So the LRT

$$\text{Reject } H_0 \text{ iff } \text{LR}(X_1, \ldots, X_n) < c$$

is equivalent to the test

$$\text{Reject } H_0 \text{ iff } \bar{X}_n > c^*$$

for some c^*.

(h) Use that fact that $\bar{X}_n \sim \text{Gamma}(n, \lambda/n)$ to find the critical value c^* in part (g) such that the test has size α.

Letting γ represent the power function of the test, the size is given by

$$\gamma(\lambda_0) = P_{\lambda_0}(\bar{X}_n > c), \quad \bar{X}_n \sim \text{Gamma}(n, \lambda_0/n),$$

so we can take c to be the upper α quantile of the $\text{Gamma}(n, \lambda_0/n)$ distribution.

(i) Under $n = 20$ and $\lambda_0 = 5$,

i. give the critical value c^* in part (g) such that the test has size $\alpha = 0.05$.

> We should take the upper 0.05 quantile of the $\text{Gamma}(20, 5/20)$ distribution, which is `qgamma(.95,shape = 20,scale=5/20)` $= 6.96981$.

ii. give the p-value corresponding to the observation $\bar{X}_{20} = 5.7$.

> This would be
>
> $$P_{\lambda_0}(\bar{X}_n > 5.7) = 1 - \texttt{pgamma(5.7,shape = 20,scale=5/20)} = 0.2505211.$$