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Basics of sets, probability

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Basics of sets

We begin by defining a *statistical experiment*.

Definition: Statistical experiment and sample space

A *statistical experiment* is a process which generates a single outcome, where:

1. There is more than one possible outcome.
2. It is known in advance what the possible outcomes are.
3. The outcome to be generated cannot be predicted with certainty.

The *sample space* S is the set of all possible outcomes.

We may refer to each possible outcome in a statistical experiment as a *sample point*.

Example. Rolling two dice and recording the rolls is a statistical experiment. We can write each possible outcome in the form (roll 1, roll 2). The sample space is

$$S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}.$$

Example. Rolling two dice and summing the rolls is a statistical experiment. The outcome is the sum of the rolls. The sample space, that is the set of all sample points, is

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

Example. Filling your car with gas and dividing the miles driven until the next fill-up by the number of gallons needed to refill the tank is a statistical experiment. The outcome is the miles-per-gallon since the last fill-up. The sample space consists of all values equal to zero or greater (think about it!):

$$S = [0, \infty).$$

Events

Definition: An event

An *event* is a statement about the outcome of a statistical experiment; it is subset of the sample space S.

Example. Roll two dice and record the rolls. An event of possible interest is "two sixes are rolled."

Example. Roll two dice and sum the rolls. An event of possible interest is "the sum of the rolls is greater than 7".

Example. Fill your car with gas and divide the miles driven until the next fill-up by the number of gallons needed to refill the tank. An event of possible interest is "the gas mileage since the last fill-up exceeds 30 mpg".

We often denote events by capital letters like A, B, and C.

Example. Roll two dice and record the rolls. Let A be the event that two sixes are rolled.

We may identify events with collections of sample points in the sample space. For example, in the experiment in which two dice are rolled we can use

- A is the event that the sum of the rolls is greater than ten
- $A = \{(5, 6), (6, 5), (6, 6)\}$

interchangeably.

Containment and equality of events

For events A and B, we say that A is *contained* in B if every point in A is in B. We write $A \subset B$. It means the same thing if we say that A is a subset of B.

We write $A = B$ if $A \subset B$ and $B \subset A$. That is, equality of sets is the property that each set is contained in the other.

The *complement* A^c of A is the event that A does *not* occur. Formally:

Definition: The complement set

The *complement* A^c of the event A is the set of all points in S which are not in A. We may write

$$A^c = S \setminus A,$$

where $S \setminus A$ means "every point in S which is not in A".

Example. Roll two dice and record the rolls. Let A be the event that two sixes are rolled. Then the event A^c is the event that the two rolls are anything other than two sixes.

Example. Roll two dice and record the rolls. Let A be the event that the sum of the rolls is greater than 7. Then A^c is the event that the sum of the rolls is less than or equal to 7. We may also write

$$A^c = \left\{ \begin{array}{llllll} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \\ (3,1) & (3,2) & (3,3) & (3,4) & & \\ (4,1) & (4,2) & (4,3) & & & \\ (5,1) & (5,2) & & & & \\ (6,1) & & & & & \end{array} \right\}.$$

If the statistical experiment is flipping a coin and recording whether it is heads or tails, the complement of the event that heads is observed does not include "an elephant walks into the room," because this is not an outcome in the sample space $S = \{\text{heads}, \text{tails}\}$. The complement of the event that heads is observed is the event that tails is observed.

Example. Roll two dice and record the rolls. Let A be the collection of pairs of rolls such that the sum of the two rolls is equal to 7. Then

$$A = \left\{ \begin{array}{cccccc} & & & & & (1,6) \\ & & & & (2,5) & \\ & & & (3,4) & & \\ & & (4,3) & & & \\ & (5,2) & & & & \\ (6,1) & & & & & \end{array} \right\}.$$

and

$$A^c = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \\ (2,1) & (2,2) & (2,3) & (2,4) & & (2,6) \\ (3,1) & (3,2) & (3,3) & & (3,5) & (3,6) \\ (4,1) & (4,2) & & (4,4) & (4,5) & (4,6) \\ (5,1) & & (5,3) & (5,4) & (5,5) & (5,6) \\ & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}.$$

Example. Fill your car with gas and divide the miles driven until the next fill-up by the number of gallons needed to refill the tank. Let A be the event that the miles-per-gallon since the last full-up exceeds 30 mpg. Then

$$A = (30, \infty) \text{ and } A^c = [0, 30].$$

Unions and intersections

Two set operations that we use very often to make new sets are the *union* and the *intersection*.

Definition: Union set

The *union* $A \cup B$ of two events A and B is the set of sample points in A or in B or in both A and B.

Definition: Intersection set

The *intersection* $A \cap B$ of two events A and B is the set of sample points in both A and B.

Example. Suppose you roll two dice and define the events

$$A = \{\text{at least one of the rolls is odd}\}$$
$$B = \{\text{at least one of the rolls is even}\}$$

Then the points in $A \cup B$ are

$$A \cup B = S,$$

and the points in $A \cap B$ are

$$\begin{aligned} A \cup B &= \{\text{One roll is odd and one roll is even}\} \\ &= \left\{ \begin{matrix} & (1,2) & & (1,4) & & (1,6) \\ (2,1) & & (2,3) & & (2,5) & \\ & (3,2) & & (3,4) & & (3,6) \\ (4,1) & & (4,3) & & (4,5) & \\ & (5,2) & & (5,4) & & (5,6) \\ (6,1) & & (6,3) & & (6,5) & \end{matrix} \right\}. \end{aligned}$$

Example. Consider an experiment in which you take two flights and record for each one whether it was on time or delayed. Then the sample space is

$$S = \{(\text{On time}, \text{On time}), (\text{Delayed}, \text{On time}), (\text{On time}, \text{Delayed}), (\text{Delayed}, \text{Delayed})\}.$$

Let

$$\begin{aligned} A &= \text{your first flight is delayed} \\ B &= \text{your second flight is delayed} \end{aligned}$$

Then

$$A \cup B = \{(\text{Delayed}, \text{On time}), (\text{On time}, \text{Delayed}), (\text{Delayed}, \text{Delayed})\}$$

is the event that there is a delay of one or the other or both of your flights and

$$A \cap B = \{(\text{Delayed}, \text{Delayed})\}.$$

is the event that both of your flights are delayed.

Example. Suppose you buy a used car and let A be the event that the transmission needs to replaced in the next year and let B be the event that the alternator needs to be replaced in the next year. Then $A \cup B$ is the probability that either your transmission or your alternator will need to be replaced in the next year, and $A \cap B$ is the event that both your transmission and your alternator will need to be replaced in the next year.

Mutually exclusive events

If two events A and B cannot both occur, they are said to be *mutually exclusive*; each one excludes the other. We give mutual exclusivity the following formal definition:

Definition: Mutually exclusive events

The events A and B are called *mutually exclusive* if

$$A \cap B = \emptyset,$$

where $\emptyset$ is called the "empty set" and is a set with zero elements.

Example. Roll a die and record the roll. Let A be the event that the roll is 2 or less and let B be the event that the roll is 5 or more, that is $A = \{1, 2\}$ and $B = \{5, 6\}$. Then

$$A \cap B = \{1, 2\} \cap \{5, 6\} = \emptyset,$$

so A and B are mutually exclusive.

The following results can be very handy. They are called De Morgan's laws:

Theorem: De Morgan's Laws

For any events A and B,

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c.$$

Exercise. A birthday party might have cake, C, and it might have ice cream, I. Interpret in words the events

1. $(C \cap I)^c$
2. $(C \cup I)^c$

Make use of De Morgan's laws if helpful.

Basics of probability

We are interested in the probabilities of events. For an event A we denote by $P(A)$ the probability that the event A occurs.

Example. Roll two dice and record the rolls. Let A be the event that two sixes are rolled. Then $P(A)$ represents the probability that two sixes are rolled.

Example. Roll two dice and let B be the event that the sum of the rolls exceeds 7. Then $P(B)$ represents the probability that the the sum of the rolls exceeds 7.

Example. Fill your car with gas and let C be the event that the miles driven until the next fill-up divided by the number of gallons needed to refill the tank exceeds 30. Then $P(C)$ is the probability that your mpg exceeds 30.

We may also simply write a statement within the $P()$.

Example. Flip two coins. $P(\text{two heads are flipped})$ represents the probability of flipping two heads.

The first rule of probability we will learn is that the probability of observing any of the possible outcomes of a statistical experiment is 1. In other words, the statistical experiment yields an outcome in the sample space with probability 1. In still other words, we always observe one of the outcomes in the sample space. Formally:

Probability rule 1: Unity of sample space probability

For a statistical experiment with the sample space S, $P(S) = 1$.

Example. Roll one die and record the roll. For this statistical experiment, the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

We can observe no value apart from the values in S—unless we are rolling the die on a funny multi-leaved table with a crevice in the middle in which the die can come to rest at an angle. Excepting such a scenario, we have

$$P(S) = P(\{1, 2, 3, 4, 5, 6\}) = 1.$$

Next, we learn that the probability of the complement of an event is one minus the probability of the event:

Probability rule 2: Probability of the complement

For an event A with complement A^c, we have

$$P(A^c) = 1 - P(A).$$

Example. Roll one die and record the roll. Let $A = \{1, 2\}$. Then $P(A) = 1/3$ and

$$P(A^c) = 1 - P(\{1, 2\}) = 2/3.$$

We can also compute $P(A^c)$ directly by noting that

$$A^c = \{1,2,3,4,5,6\} \setminus \{1,2\} = \{3,4,5,6\},$$

and

$$P(\{3,4,5,6\}) = 2/3.$$

Probability rule 3: Nullity of empty-set probability

The probability of the empty set $\emptyset$ is $P(\emptyset) = 0$.

Note that the complement S^c of the sample space S is the empty set $\emptyset$, and we can see that

$$P(S^c) = 1 - P(S) = 0,$$

since $P(S) = 1$.

Now we consider the probability of a union. We start with the following result:

Probability rule 4: Probability of a union of two events

For any two events A and B we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Draw a Venn Diagram to visualize this!

Example. Let A be the event that you get the flu this semester and let B be the event that one of your roommates gets the flu this semester. Suppose that $P(A) = 0.03$ and $P(B) = 0.05$, and suppose that $A \cap B$, the event that both both you and one of your roommates gets the flu this semester, has probability $P(A \cap B) = 0.025$. Then the probability of the event $A \cup B$ is

$$P(A \cup B) = 0.03 + 0.05 - 0.025 = 0.055$$

Example. Suppose friend 1 of yours shows up to parties with probability 0.4 (call this event F_1) and friend 2 of yours shows up with probability 0.5 (call this event F_2).

1. Suppose that these friends don't like each other, so that the probability that they both show up at a party is .01. What is probability that either of your friends comes to the

party?

$$\begin{aligned} P(\text{either friend at party}) &= P(F_1) + P(F_2) - P(F_1 \cap F_2) \\ &= 0.4 + 0.5 - 0.01 \\ &= 0.89 \end{aligned}$$

2. What is probability that neither of your friends comes to the party?

$$\begin{aligned} P(\text{neither friend at party}) &= P((F_1 \cup F_2)^c) \\ &= 1 - P(F_1 \cup F_2) \\ &= 1 - 0.89 \\ &= 0.11 \end{aligned}$$

3. Now suppose your friends really like each other, so that the probability that they both show up to a party is 0.35. What is now the probability that either of your friends comes to the party?

$$\begin{aligned} P(\text{either friend at party}) &= P(F_1) + P(F_2) - P(F_1 \cap F_2) \\ &= 0.4 + 0.5 - 0.35 \\ &= 0.55 \end{aligned}$$

4. Why does this make sense? In part 1, your friends tend not to go to the same parties, so you are more likely to bump into one of them if you choose a random party. In part 2, your friends tend to go to the same parties, so you are more likely to miss them if you choose a random party.

If the events A and B are mutually exclusive, it means that $A \cap B = \emptyset$. Therefore if $A \cap B$ are mutually exclusive, $P(A \cap B) = 0$. The probability is 0 that they both occur. As a consequence, if A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B),$$

since $P(A \cap B) = 0$. So we give the following rule:

Probability rule 5: Probability of the union of mutually exclusive events

If the events A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Example. Suppose there is some drama between friend 1 and friend 2 of yours, and for this weekend the probability that they both show up to a party is equal to zero. Then if friend 1 of yours shows up to parties with probability 0.4 and friend 2 of yours shows up with probability 0.5, what is the probability that at least one of the two friends shows up to a party?

$$P(\text{either friend at party}) = P(F_1) + P(F_2) = 0.4 + 0.5 = 0.9.$$