

STAT 515 fa 2023 Lec 01 slides

Basics of sets, probability

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

GATHERING
DATA

ANALYZE

FIND PATTERNS

RESEARCH

Statistics: What can we learn from data

— given that it can change
every time we do an experiment?

STATISTICS

PROBABILITY

SET THEORY

1 Basics of sets

2 Basics of probability

Statistical

Experiment

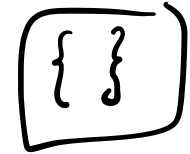
An *experiment* is a process which generates an outcome such that there is

- (i) more than one possible outcome
- (ii) the set of possible outcomes is known
- (iii) the outcome is not known in advance

Sample space and sample points

S

- The sample space S of an experiment is the set of possible outcomes.
- The outcomes in a sample space S are called sample points.



Exercise: Give the sample space S for the following experiments:

- 1 Roll of a 6-sided die. $S = \{1, 2, 3, 4, 5, 6\}$
- 2 Number of COVID-19 cases confirmed in SC tomorrow. $S = \{0, 1, 2, \dots, N\}$
p.p. SC ↓
- 3 Number of stars counted in a patch of space. $S = \{0, 1, \dots\}$
- 4 Blood type of a randomly selected student. $S = \{A, B, AB, O\}$
- 5 Time until you drop your ~~new~~ phone and crack the screen. $S = [0, \infty)$
- 6 Proportion of people in a sample with antibodies to a virus. $S = [0, 1]$
- 7 Deviation of today's temperature from the historical average.

$$S = (-\infty, \infty)$$

Event

An *event* is a collection of possible outcomes of an experiment, that is any subset of S (including S itself).

- Usually represent events with capital letters A, B, C, \dots
- Say an event A occurs if the outcome is in the set A .
- So events are equivalent to sets. Can refer to events as sets, to sets as events.
- Often refer to members of sets as *elements* of the set.

Exercise: Express the following events as subsets of the sample space.

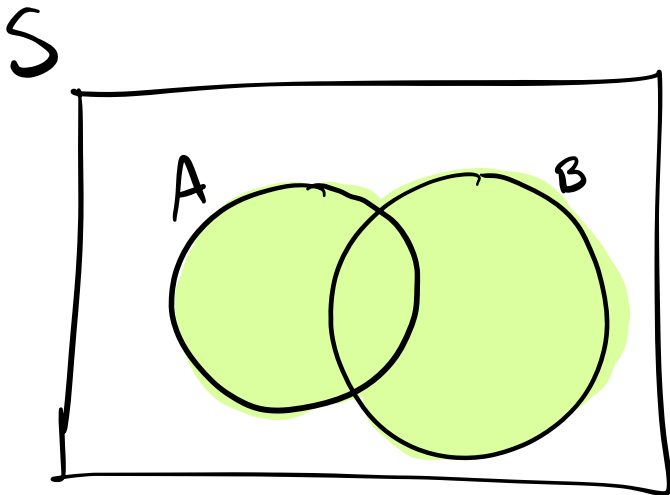
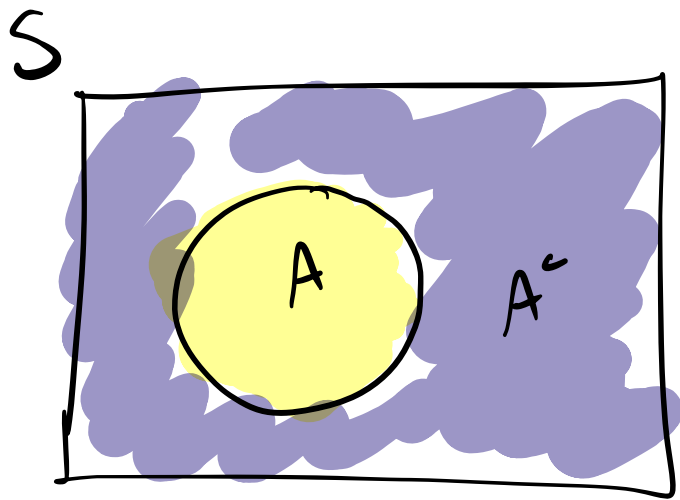
- 1 Roll of a 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
 $A = \text{odd number rolled.}$ $A = \{1, 3, 5\}$
- 2 Number of COVID-19 cases confirmed in SC tomorrow:
 $A = \text{no new cases.}$ $A = \{0\}$ "singleton"
- 3 Number of stars counted in a patch of space:
 $A = \text{more than 1,000.}$ $A = \{1001, 1002, \dots\}$
- 4 Blood type of a randomly selected student:
 $A = \text{has } \underline{A} \text{ antigen in RBCs.}$ $A = \{A, AB\}$
- 5 Time until you drop your new phone and crack the screen: $S = [0, \infty)$
 $A = \text{within the first month.}$ $A = [0, 1)$
- 6 Proportion of people in a sample with antibodies to a virus:
 $A = \text{60\% or more with antibodies.}$ $A = [0.60, 1]$
- 7 Deviation of today's temperature from the historical average:
 $A = \text{at least 10 degrees cooler.}$ $A = (-\infty, -10]$ $S = (-\infty, \infty).$

Elementary set operations

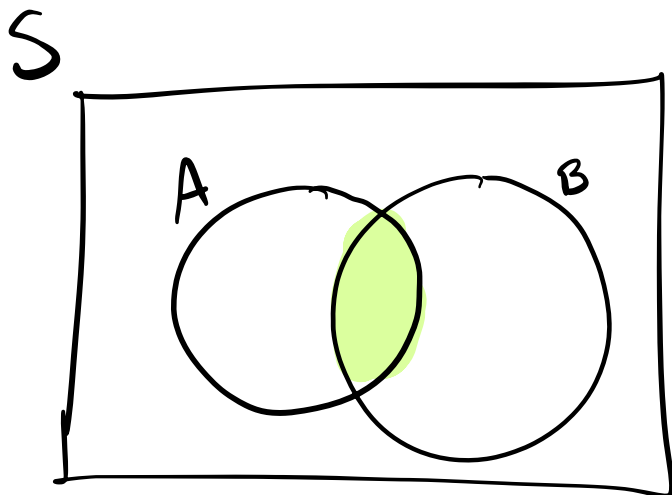
- The *complement set* : The complement A^c of the event A is all the outcomes in S which are not in A .
- The *union set*
Let A and B be events in S . The union $A \cup B$ the outcomes in A or B or in both A and B .
- The *intersection set*

The intersection $A \cap B$ is the outcomes in both A and B .

Draw pictures.



$A \cup B$

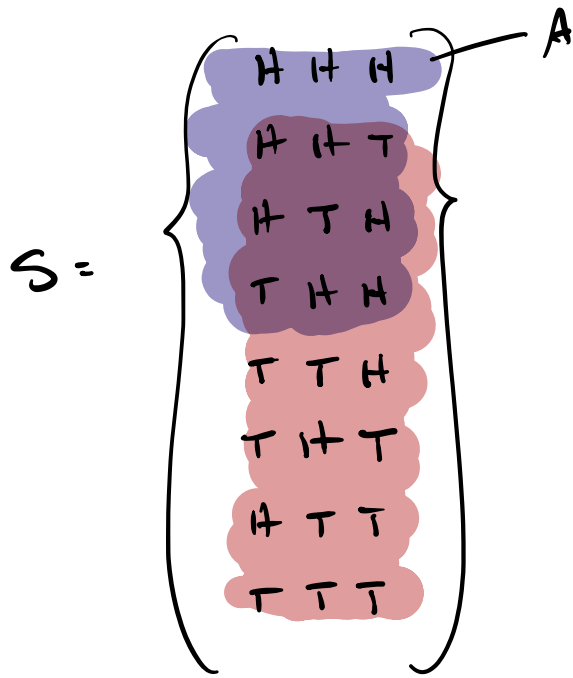


$A \cap B$

Exercise: Suppose we flip a coin three times and record the sequence of heads and tails.

- 1 Give the sample space S .
- 2 Give the points in the event $A =$ at least two heads come up.
- 3 Give the points in the event $B =$ ~~exactly two tails come up.~~
at least one tails
- 4 Give the points in the event $B \cap A$.
- 5 Give the points in the event $B^c \cup A^c$. ✓
- 6 Give the points in the event $(B \cup A)^c$.
- 7 Give the points in the event $(B \cap A)^c$.
- 8 Give the points in A^c .

Flip a coin 3x



$$A = \text{At least 2 heads}$$

$$B = \text{At least 2 tails}$$

$$A \cap B = \{ HHT, HTH, THH \}$$

$$B^c = \{ HHH \}$$

$$A^c = \{ TTH, THT, HTT, TTT \}$$

$$B^c \cup A^c = \{ TTH, THT, HTT, TTT, HHH \}$$

$$(B \cup A)^c = \emptyset \leftarrow \text{"Empty set"}$$

$$(B \cap A)^c = \{ TTH, THT, HTT, TTT, HHH \} \stackrel{\text{wow!}}{=} (B \cup A)^c$$

Exercise: Suppose we roll a 6-sided die twice. The sample space is

$$S = \left\{ \begin{array}{cccccc} (1, 1) & (1, 2) & (1, 3) & (1, 4) & (1, 5) & (1, 6) \\ (2, 1) & (2, 2) & (2, 3) & (2, 4) & (2, 5) & (2, 6) \\ (3, 1) & (3, 2) & (3, 3) & (3, 4) & (3, 5) & (3, 6) \\ (4, 1) & (4, 2) & (4, 3) & (4, 4) & (4, 5) & (4, 6) \\ (5, 1) & (5, 2) & (5, 3) & (5, 4) & (5, 5) & (5, 6) \\ (6, 1) & (6, 2) & (6, 3) & (6, 4) & (6, 5) & (6, 6) \end{array} \right\}.$$

Define the events

$$A = \{\text{at least one of the rolls is odd}\}$$

$$B = \{\text{sum of the rolls is a prime number}\}$$

$$C = \{\text{sum of the rolls is an even number}\}$$

$$D = \{\text{Both rolls are even numbers}\}$$

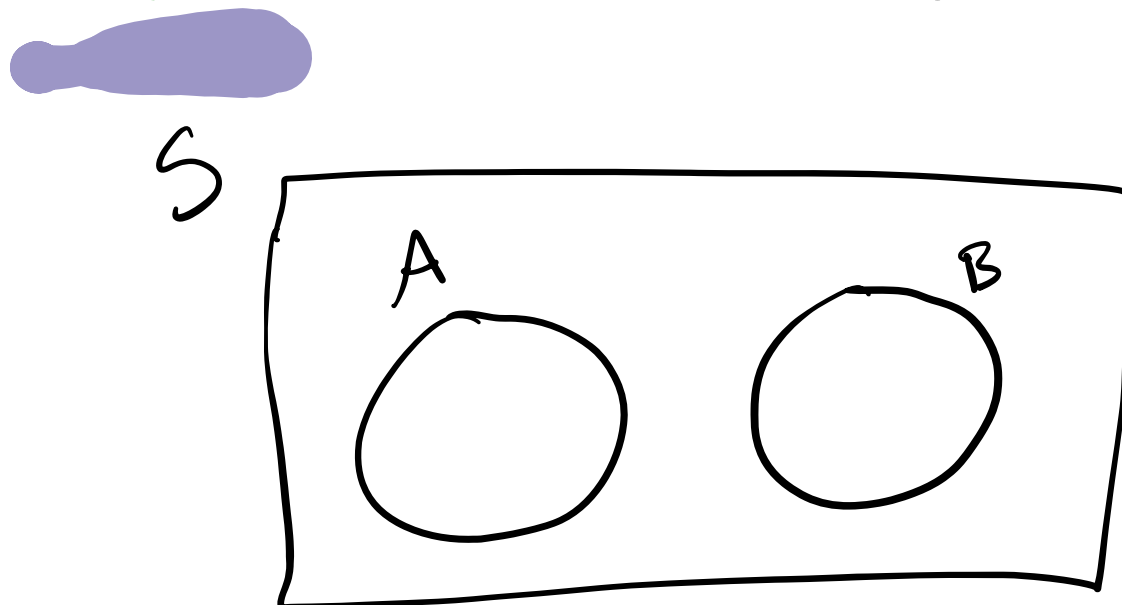
- 1 Give $A \cup B$.
- 2 Give $A \cap B$.
- 3 Give $B \cap C$.

$$A \cap D = \emptyset$$

Mutual exclusivity/disjoint-ness

Two events A and B are called *mutually exclusive* or *disjoint* if $A \cap B = \emptyset$.

The set \emptyset is the the *empty set*, which is the set containing no elements.



De Morgan's Laws

For any events A and B we have

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c$$

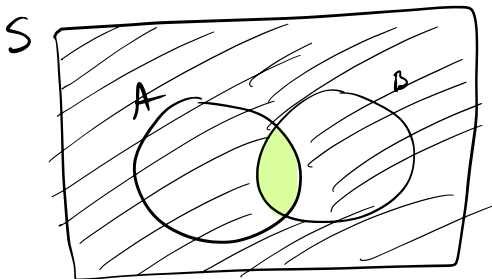
Draw pictures.

Exercise: A birthday party might have cake C and it might have ice cream I . Interpret in words the events

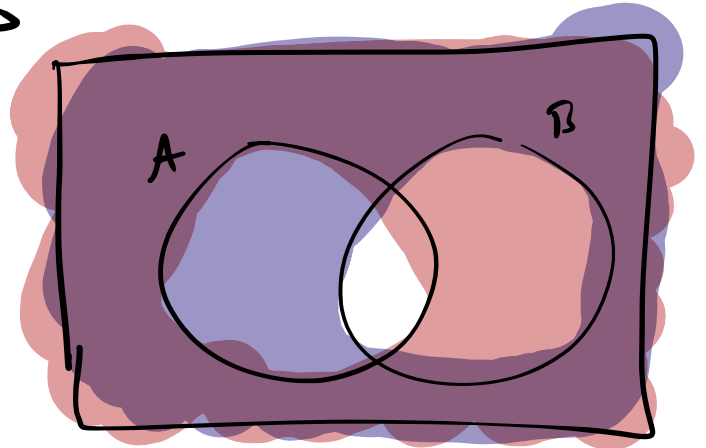
- 1 $(C \cap I)^c$
- 2 $(C \cup I)^c$

De Morgan's Laws

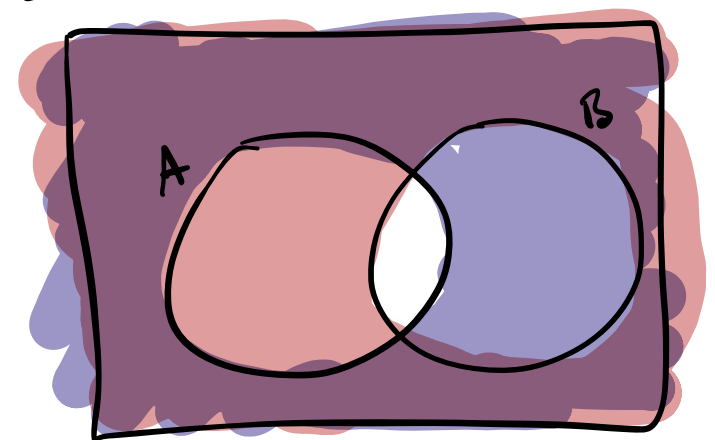
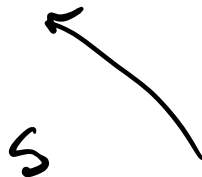
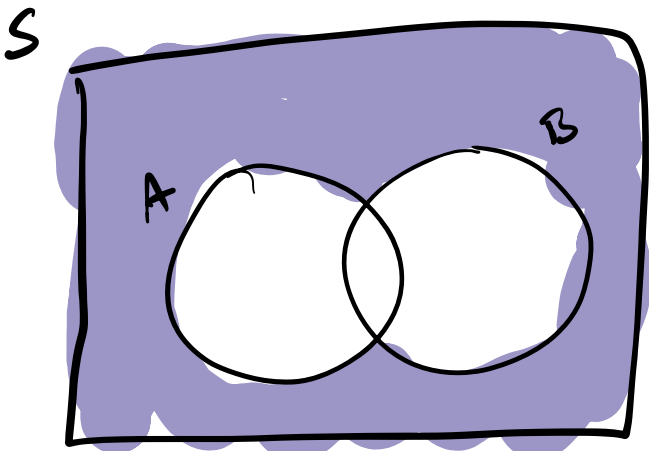
$$(A \cap B)^c = A^c \cup B^c$$



=



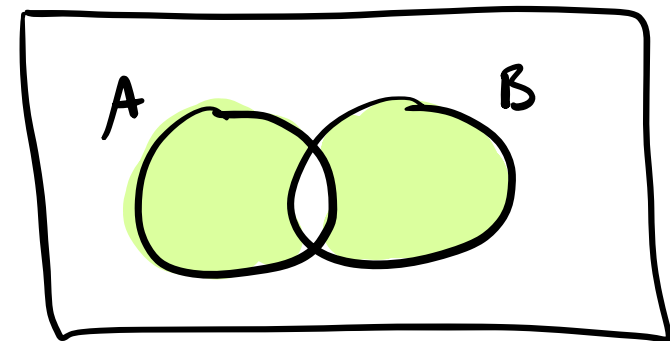
$$(A \cup B)^c = A^c \cap B^c$$



Exercise: For two events A and B use our elementary set operations \cup , \cap , and the complement to give representations of the following:

- Both A and B occur. $A \cap B$
- Neither A nor B occurs; give two representations of this. $A^c \cap B^c = (A \cup B)^c$
- At least one of the events A and B occurs. $A \cup B$
- At least one of the events A and B does not occur; give two representations of this. $A^c \cup B^c = (A \cap B)^c$
- One of the events A and B occurs but not the other.

$$(A \cap B^c) \cup (B \cap A^c)$$



Exercise: Suppose a wildebeest W , a crocodile C , and a giraffe G on a safari. Write down the following events using elementary set operations. You see...

- 1 a giraffe ^{and not} but no wildebeest. $G \cap W^c$
- 2 all three types of animals. $G \cap W \cap C$
- 3 not all three types of animals. $(G \cap W \cap C)^c$
- 4 a giraffe and a wildebeest but no crocodile. $G \cap W \cap C^c$
- 5 not both a giraffe and a wildebeest. $(G \cap W)^c$
- 6 a giraffe and a wildebeest without seeing a crocodile or a crocodile and a wildebeest without seeing a giraffe. $(G \cap W \cap C^c) \cup (C \cap W \cap G^c)$
- 7 ~~any~~ ^{exactly} two of the three types of animal. $(G \cap W \cap C^c) \cup (C \cap W \cap G^c) \cup (C \cap W^c \cap G)$
- 8 exactly one of the three types of animal.
- 9 at least one of the three types of animal. $G \cup W \cup C$
- 10 none of the animals.

$$(G \cup W \cup C)^c$$

④

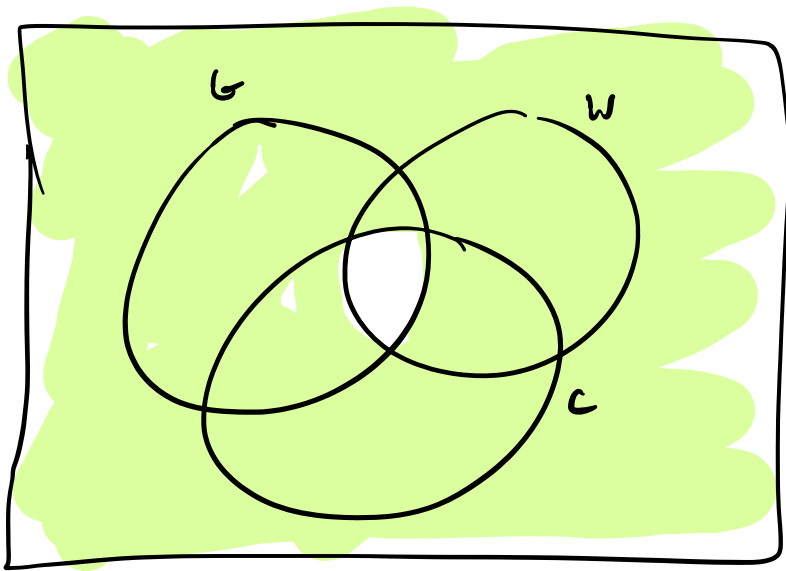
$$(G \cap W) \cup (C \cap W) \cup (C \cap G)$$

⑧

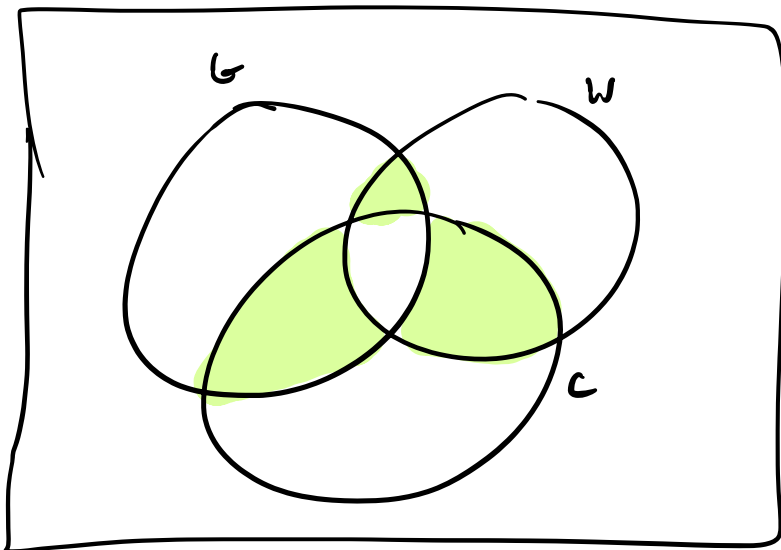
De Morgan's

$$\begin{aligned} ((G \cap W) \cap C)^c &= (G \cap W)^c \cup C^c \\ &= G^c \cup W^c \cup C^c \end{aligned}$$

$(C \cap G \cap W)^c$

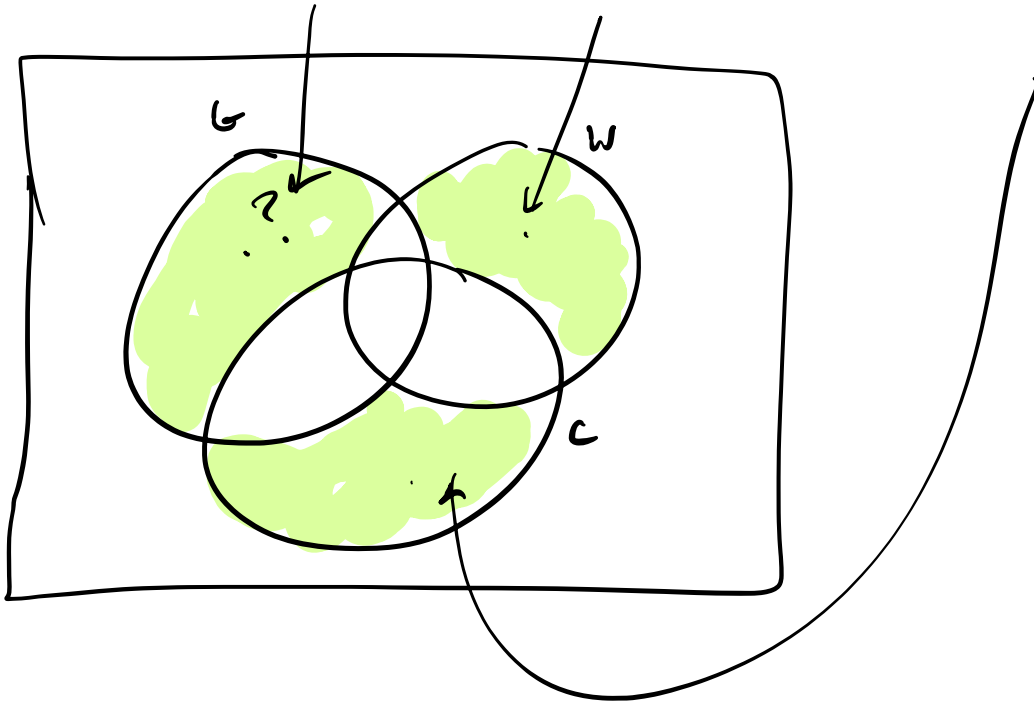


$$(G \cap W \cap C^c) \cup (C \cap W \cap G^c) \cup (C \cap W^c \cap G)$$



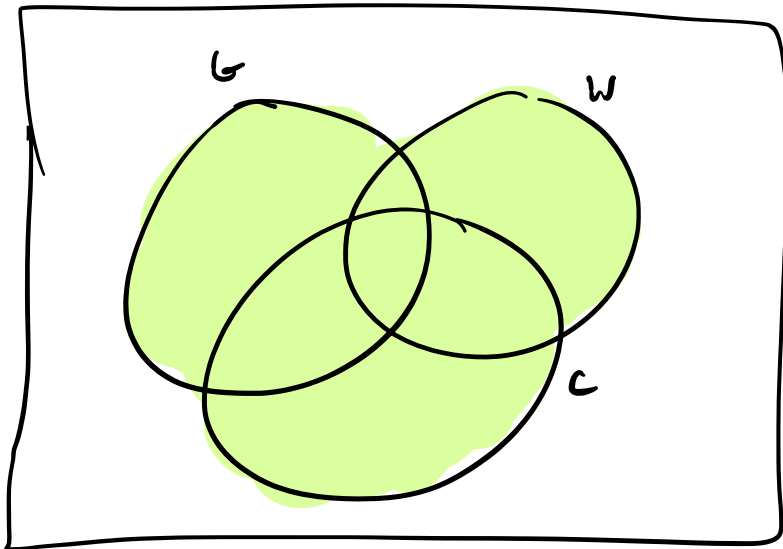
(8)

$$(G \cap W^c \cap C^c) \cup (W \cap G^c \cap C^c) \cup (C \cap G^c \cap W^c)$$



(9)

$$G \cup C \cup W$$



1 Basics of sets

2 Basics of probability




$$P(A) = \text{number}$$

Probability notation

For an event A , we denote by $P(A)$ the probability that A occurs.

While A is an event or a set, $P(A)$ is a number in the interval $[0, 1]$.

Examples:

- 1 Roll a die and let A be the event that  is rolled. Then $P(A) = 1/6$.
- 2 Roll two dice and let B be the event that ,  is rolled. Then $P(B) = 1/36$.
- 3 Let C be the event that you get to park in your favorite spot. Then $P(C) = \text{some number}$.

Now some rules about probabilities...

$$\underline{\text{Axiom 1:}} \quad P(A) \geq 0$$

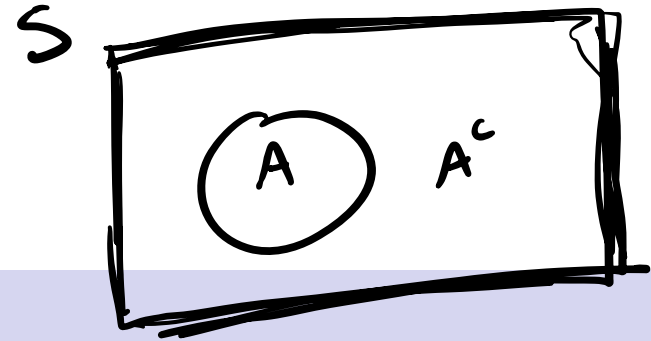
$$\underline{\text{Axiom 2:}}$$

Unity of the sample space probability

For a statistical experiment with sample space S , $P(S) = 1$.

Examples:

- Rolling a die: $S = \{\square, \blacksquare, \blacklozenge, \blacktriangle, \blacktriangledown, \blacktriangleright\}$. We have $P(S) = 1$.
- MPG on next tank of gas: $S = [0, \infty)$. We have $P(S) = 1$.
- # students in class of 40 with cloth masks: $S = \{0, 1, \dots, 40\}$. $P(S) = 1$.

Axiom 3:

Probability of a complement

For an event A with complement A^c, $P(A^c) = 1 - P(A)$.

Examples:

- Roll a die and let $A = \{\square, \circ\}$. Then $P(A) = 1/3$. Give $P(A^c)$. $P(A^c) = 2/3$.
- We have $P(\emptyset) = 0$. Why?

$$P(S) = 1 \quad S^c = \emptyset$$

$$P(S^c) = 1 - P(S) = 1 - 1 = 0.$$

Probability of a union

For any two events A and B , we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Draw a picture...

Exercise: Let F_1 and F_2 be the events that you meet friend 1 and friend 2 at a party, respectively. Suppose $P(F_1) = 0.4$, $P(F_2) = 0.5$ and $P(F_1 \cap F_2) = 0.01$.

Find the probabilities of the following:

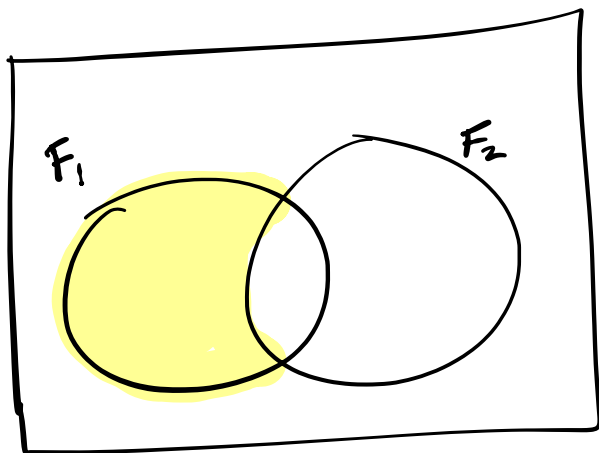
- ① You meet at least one of the two friends at a party.
- ② You do not meet either of your friends at a party.
- ③ You meet friend 1 but not friend 2 at a party.

$$\textcircled{1} \quad P(\underline{F_1 \cup F_2}) = P(F_1) + P(F_2) - P(F_1 \cap F_2) = 0.4 + 0.5 - 0.01 = .89.$$

$$\textcircled{2} \quad P((F_1 \cup F_2)^c) = 1 - P(F_1 \cup F_2) = 1 - 0.89 = 0.11$$

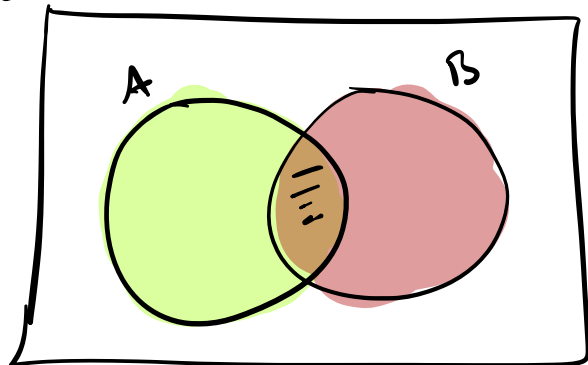
$$\begin{aligned} \textcircled{3} \quad P(F_1 \cap F_2^c) &= P(F_1) - P(F_1 \cap F_2) \\ &= 0.40 - 0.01 \\ &= 0.39 \end{aligned}$$

S



$$P(A \cup B) = \underline{P(A)} + \underline{P(B)} - \underline{P(A \cap B)}$$

S



$$\begin{array}{c} \downarrow \quad \downarrow \\ P(A) \cup P(B) \end{array}$$

$$A \cap B = \emptyset$$

Always: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$A \cap B = \emptyset$$



Probability of the union of mutually exclusive events

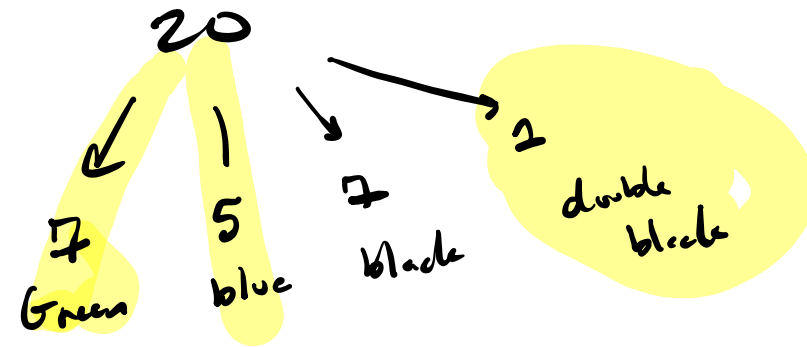
If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{=0}$$

Exercise: You choose a ski slope at random from among 20 slopes, with 7 marked with a green circle, 5 with a blue square, 7 with a black diamond, and 1 with double black diamonds. Give the probability that you choose a slope marked with

- 1 a green circle.
- 2 a green circle or a blue square.
- 3 anything but double black diamonds.

$$\textcircled{1} P(G) = \frac{7}{20}$$



$$\textcircled{2} \quad P(A \cup B) = P(A) + P(B) = \frac{7}{20} + \frac{5}{20} = \frac{12}{20}.$$

$$\begin{aligned} \textcircled{3} \quad P(\text{double Black})^c &= 1 - P(\text{double black}) \\ &= 1 - \frac{1}{20} \\ &= \frac{19}{20}. \end{aligned}$$

In some situations we can compute probabilities by counting sample points.

Computing probabilities when all sample points equally likely

If all outcomes in S are equally likely, for any event $A \subset S$, we have

$$P(A) = \frac{\#\{\text{sample points in } A\}}{\#\{\text{sample points in } S\}}$$

only works if finite # sample points

Exercise: Suppose we roll two 6-sided dice. Find

① $P(\text{sum of rolls equals } 7) = \frac{6}{36} = \frac{1}{6}$

② $P(\text{we roll doubles}) = \frac{6}{36} = \frac{1}{6}$

③ $P(\text{sum of rolls at least } 10) = \frac{6}{36} = \frac{1}{6}$

④ $P(\text{at least one roll is greater than } 3) = \frac{27}{36} = \frac{3}{4}$

{ sample points in S } = 36

$$S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}.$$

vents

Exercise: A jukebox will play a song at random from a library of 50 songs, of which 20 are R&B, 10 are country, 12 are pop, and 8 are rock. Of the R&B songs, 15 are less than a decade old, and of each other genre, half of the songs are less than a decade old. Find the probability that the jukebox plays

- 1 A pop song.
- 2 An R&B song less than a decade old.
- 3 A song less than a decade old.
- 4 A rock song more than a decade old.
- 5 A country song or any song that is more than a decade old.

	R&B	Country	pop	rock	total
D less than decade old	15	5	6	4	30
D ^c older than 10 yrs	5	5	6	4	20
total	20	10	12	8	50

$$\textcircled{1} P(\text{pop}) = \frac{12}{50}$$

$$\textcircled{2} P(\text{R\&B} \cap (\text{less than decade old})) = \frac{15}{50}$$

$$\textcircled{3} P(\text{less than decade old}) = \frac{30}{50}$$

$$\textcircled{4} P(\text{Rock} \cap \text{more than 10yr}) = \frac{4}{50}$$

$$\textcircled{5} P(\text{Country} \cup \text{decade old}) = \frac{25}{50}$$