# STAT 515 fa 2023 Lec 02 slides

## Counting rules

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

$\subset$ : "subset"

## Motivation to study counting rules

If all outcomes in $S$ are equally likely, for any event $A \subset S$, we have

in

$$P(A) = \frac{\#\{\text{sample points in } A\}}{\#\{\text{sample points in } S\}}.$$

This leads to an interest in counting rules.

## Fundamental theorem of counting

If a $\boxed{\text{job}}$ consists of $K$ tasks such that the tasks may be completed in $n_1, \ldots, n_K$ ways, respectively, then there are

product symbol

$$\prod_{k=1}^{K} n_k = n_1 \times n_2 \times \cdots \times n_K$$

ways to do the job.

**Exercise:** You are confronted with the following sequence of choices:

$K = 5$

1. Barbacoa, chicken, carnitas, or veggies $n_1 = 4$
2. White or brown rice $n_2 = 2$
3. Black or pinto beans $n_3 = 2$
4. Spicy, medium, or mild $n_4 = 3$
5. To pay extra for guacamole or not to pay extra for guacamole $n_5 = 2$

In which restaurant are you ordering food?

Total # burritos possible = $\prod_{k=1}^{5} n_k = n_1 \cdot n_2 \cdot n_3 \cdot n_4 \cdot n_5$

$= 4 \cdot 2 \cdot 2 \cdot 3 \cdot 2$

$= 96$

$7! = $ "7 factorial" $= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

## Permutation

Draw $r$ elements of $N$ elements without replacement and arrange them in some order. The # ways is

$$N(N-1)\cdots(N-r+1) = \frac{N!}{(N-r)!}. = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 \cdot 5 \cdot 4$$

4

(1) $7 \cdot 6 \cdot 5 \cdot 4$

$N \quad N-1 \quad N-2 \quad N-3 = N-4+1$

**Exercise:** You must compose a ballad in the key of G, i.e. from the chords

G Am Bm C D Em F#dim

$N = 7$

$r = 4$

1. In how many ways can four distinct chords be chosen to begin your song?
2. What if you can't play F#dim?

$K=4$

| | | number of ways | |
|---|---|---|---|
| Task 1 : | choose 1st chord | 7 | 6 |
| Task 2 : | choose 2nd chord | 6 | 5 |
| Task 3 : | choose 3rd chord | 5 | 4 |
| Task 4 : | choose 4th chord | 4 | 3 |

(1) # ways = $7 \cdot 6 \cdot 5 \cdot 4 = 42 \cdot 20 = 840$

(2) # ways = $6 \cdot 5 \cdot 4 \cdot 3 = 30 \cdot 12 = 360$

# ways to read 2 of the 6 novels in any order is

$$\binom{6}{2} = \frac{6!}{2!\,(6-2)!} = \frac{6!}{2!\,4!} = \binom{6}{4}$$

Draw $r$ from $N$ things....

**Exercise:** Jane Austen wrote the following novels:

$N = 6$

$r = 6$

① 1st book : 6

choose 2nd : 5

⋮

choose 6 : 1

1 { Pride and Prejudice
  Emma
  Sense and Sensibility

2 Persuasion

3 Northanger Abbey

4 Mansfield Park

$$\# \text{ways} = \frac{N!}{(N-r)!} = \frac{6!}{(6-6)!} = \frac{6!}{0!} = 6!$$

always $0! = 1$

1. In how many ways can you read them all this semester? $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
2. In how many ways can you read them all such that you read Pride and Prejudice, Emma, and Sense and Sensibility without reading any of the others in between?

$$\# \text{ways} = 4! \cdot 3!$$

## Partition

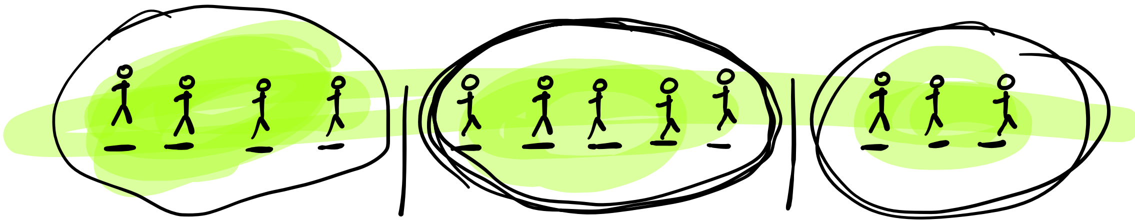
Place $N$ balls into $K$ urns such that the $K$ urns receive $n_1, \ldots, n_K$ balls, where $N = n_1 + \cdots + n_N$. The # ways is

$$\frac{N!}{n_1! n_2! \cdots n_K!}.$$

**Exercise:** Suppose 12 people are randomly assigned to ride in 3 vehicles taking 4, 5, and 3 passengers, respectively.

1. In how many ways can the passengers be assigned to the different vehicles?
2. If you and your friend are among these people, with what probability will the two of you ride in the same vehicle?

$K = 3$ $N = 12$, $n_1 = 4$, $n_2 = 5$, $n_3 = 3$



12! total seating arrangements — caring about specific seats within each vehicle

$$\# \text{ ways} = \frac{12!}{4!\ 5!\ 3!}$$ = # ways to put people in groups in the cars NOT caring about where they sit in the car.

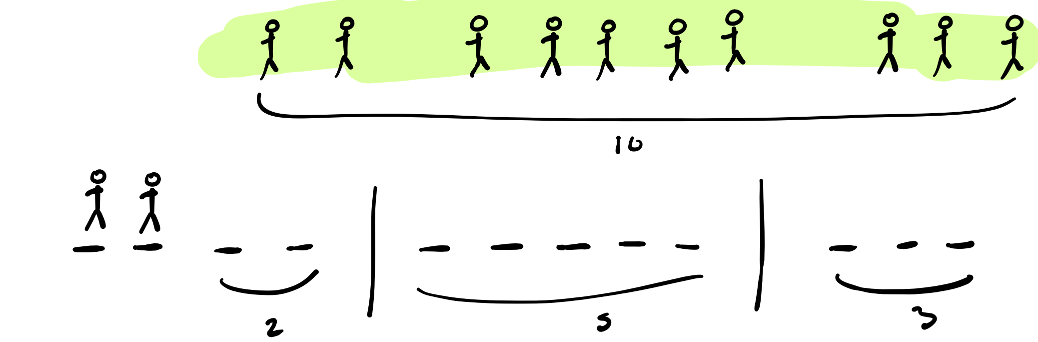
$= 27{,}720$

(2) A = Event you and friend together

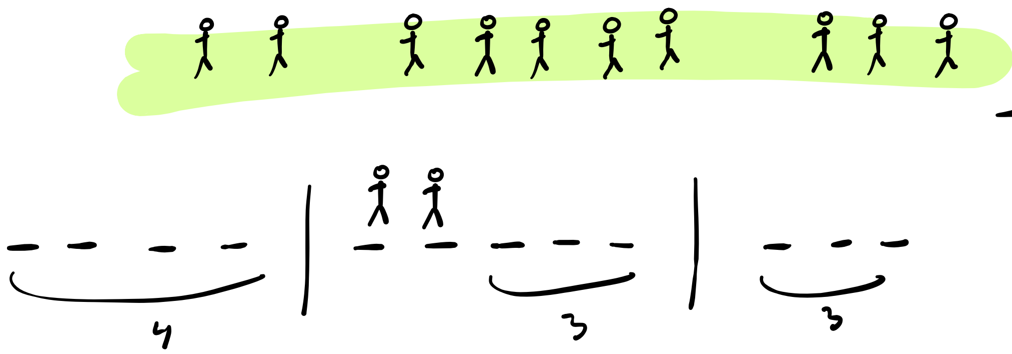
S = All ways to partition the 12 people.

$$P(A) = \frac{\#\{\text{outcomes in } A\}}{\#\{\text{outcomes in } S\}} = \frac{\quad}{27{,}720}$$

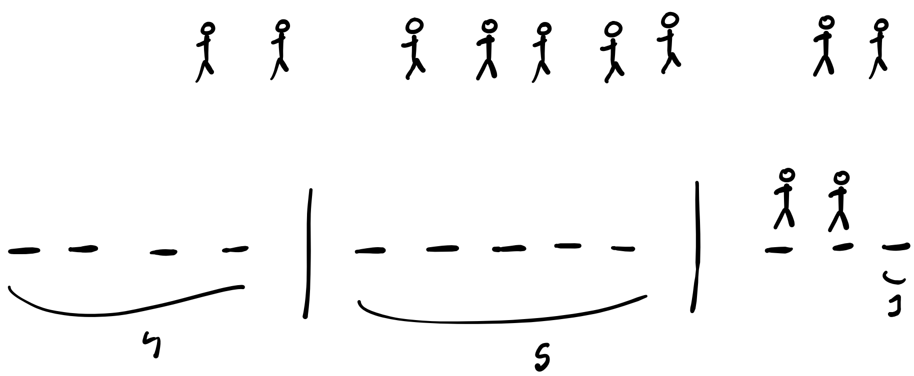
Job: Put you & friend in a vehicle together and then put everyone else in the cars




$$\frac{10!}{2!\,5!\,3!} = 2520$$




$$\frac{10!}{4!\,3!\,3!} = 4200$$




$$\frac{10!}{4!\,5!\,1!} = 1260$$

$$P\left(\text{You + friend together}\right) = \frac{7980}{27720}$$

## Combination

Draw $r$ elements from $N$ without replacement and without regard to their order. The # ways is

"N choose r"

$$\binom{N}{r} = \frac{N!}{r!(N-r)!}.$$

$$\frac{10!}{5!(10-5)!} = \frac{10!}{5!5!}$$

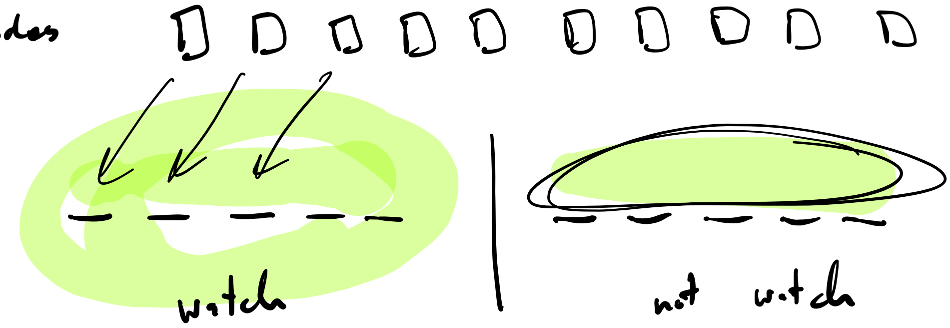
This is just a partition with only 2 "urns".

**Exercise:** For German class you must watch 5 episodes from among 10 available episodes of Bares für Rares, 8 of Betty's Diagnose, and 12 of Haustier Check.

1. In how many ways can you choose
   - 5 episodes of Bares für Rares?
   - 3 episodes of Haustier Check and 2 episodes of Betty's Diagnose?
2. If you choose 5 episodes at random, with what probability do you
   - not watch any episodes of Haustier Check?
   - watch at least one episode of Bares für Rares?
   - binge entirely on Betty's Diagnose?

② (i) # ways to choose 5 episodes of Bares für Rares from among 10 available episodes.

episodes




$$\frac{10!}{5!\,5!}$$

(ii) # ways watch 3 of Haustier Check (12 avail) and 2 of B's D. (8 avail)

$$\underset{\text{Task 1}}{\binom{12}{3}} \cdot \underset{\text{Task 2}}{\binom{8}{2}} = 6160$$

3 episodes of Haustier Check and 2 episodes of Betty's Diagnose?

2. If you choose 5 episodes at random, with what probability do you
   - (1) ▸ not watch any episodes of Haustier Check? ← 12 episodes, 18 episodes of the other two shows
   - (2) ▸ watch at least one episode of Bares für Rares?
   - (3) ▸ binge entirely on Betty's Diagnose?

$$P(A) = \frac{\#\{A\}}{\#\{S\}}$$

① 30 HM episodes.

$\binom{30}{5}$ ways to choose 5 episodes

$\binom{18}{5}$ ways to choose 5 episodes which are NOT HTC.

$$P(A) = \frac{\binom{18}{5}}{\binom{30}{5}}$$

② (Watching at least 1 episode of BfR)

$$= (\text{Watching } 0 \text{ episodes of BfR})^c$$

$$\Uparrow$$

$$P(\text{watch } 0 \text{ BfR}) = \frac{\binom{20}{5}}{\binom{30}{5}}$$

$$P(\text{watch at least } 1 \text{ of BfR}) = 1 - \frac{\binom{20}{5}}{\binom{30}{5}}$$

$$P(A) = \frac{\# A}{\# S} = \frac{\#\{5 \text{ card hands with } A\diamondsuit\}}{\#\{5 \text{ card hands}\}}$$

$$S = \{\text{All possible 5-card hands}\}$$

**Exercise:** If dealt 5 cards from a 52-card deck, what is the probability of getting

1. the ace of diamonds?
2. at least one ace? ← Complement?

$$\frac{1!}{1!(1-1)!} = 1$$

$$\#\{5 \text{ card hands with } A\diamondsuit\} = \binom{1}{1} \cdot \binom{51}{4} = \frac{51!}{4!(51-4)!}$$

$$\#\text{ 5-card hands} = \binom{52}{5} = \frac{52!}{5!(52-5)!}$$

$$\frac{\#\{5 \text{ card hands with } A\diamond\}}{\#\{5 \text{ card hands}\}} = \frac{\binom{1}{1}\cdot\binom{51}{4}}{\binom{52}{5}} = \frac{\frac{51!}{4!(51-4)!}}{\frac{52!}{5!(52-5)!}}$$

$$= \frac{\cancel{51!}}{4!\,\cancel{47!}}\;\frac{5!\,\cancel{47!}}{52!} \qquad 52\cdot\cancel{51!}$$

$$= \frac{5}{52}.$$

② $P(\text{at least 1 Ace}) = 1 - P(\text{No Aces})$

$$= 1 - \frac{\#\{5 \text{ card hands no Aces}\}}{\#\{5\text{-card hands}\}}$$

$$= 1 - \frac{\binom{48}{5}}{\binom{52}{5}}$$

$= 1 -$ choose(48, 5) / choose(52, 5).