STAT 515 fa 2023 Lec 03 slides

Conditional probability, independence, Bayes' rule

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Conditional probability

The *conditional probability* of A given B is

$$\boxed{\underline{P(A|B)} = \frac{P(A \cap B)}{P(B)}.}$$

Memorize

This is the probability that the event A occurs given that B occurs.

A

Exercise: Roll two dice. Find

B

1. $P($ doubles $)$
2. $P($ sum $\geq 10)$
3. $P($ doubles $|$ sum $\geq 10)$
4. $P($ sum $\geq 10|$ doubles $)$

$P(A) = \frac{6}{36} = \frac{1}{6}$ $\quad P(B) = \frac{6}{36} = \frac{1}{6}$

$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{2/36}{6/36} = \frac{2}{6} = 1/3$

$$\mathcal{S} = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}$$

$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{2/36}{6/36} = \frac{1}{3}$

Exercise: From STAT 515 fa 2019:

- 40 students in class
- 10 students got an 'A' on final exam
- 12 students got an 'A' hw average
- 23 students did not get an 'A' on the final or an 'A' hw average.

If a student is drawn at random from the class, give

1. $P($ 'A' on final $)$ $= 10/40 = P(A)$
2. $P($ 'A' on final $|$ 'A' hw average $)$ $= P(A|B) = \boxed{\frac{P(A \cap B)}{P(B)}} = \frac{5/40}{12/40} = \frac{5}{12}$
3. $P($ 'A' on final $|$ less than 'A' hw average $)$ $= P(A|B^c) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{5/40}{28/40} = \frac{5}{28}$
4. $P($ less than 'A' on final $|$ less than 'A' hw average $)$
5. $P($'A' on final $\cap$ 'A' hw average $)$

A = A on Final
B = A hw avg

	A	A^c	total
B	5	7	12
B^c	5	23	28
total	10	30	40

Are A and B independent?

$$P(A \cap B) \overset{?}{=} P(A)\,P(B)$$

$$\frac{5}{40} \neq \frac{10}{40}\,\frac{12}{40} = \frac{120}{1600} = \frac{12}{160} = \frac{3}{40}$$

not equal

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \iff P(A \cap B) = P(B|A)\,P(A).$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \iff P(A \cap B) = P(A|B)\,P(B)$$

Intersection prob. as conditional times unconditional

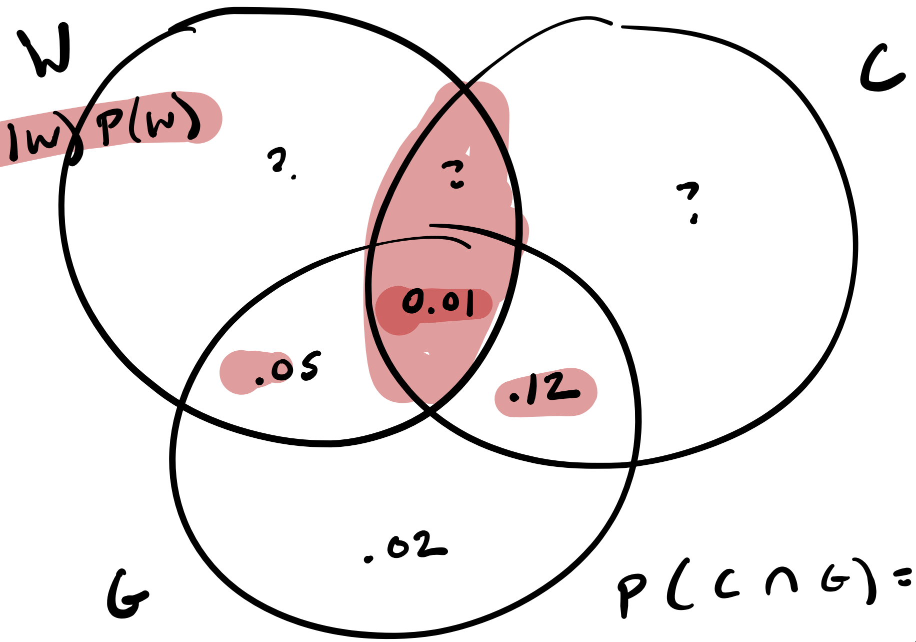
For any two events A and B,

$$P(A \cap B) = P(A|B)P(B) \quad \text{or} \quad P(A \cap B) = P(B|A)P(A).$$

Exercise: Suppose that on a safari, the probabilities of seeing a giraffe (G), a wildebeest (W), and a crocodile (C) are as follows:

$$P(W) = 0.40$$
$$P(C) = 0.60$$
$$P(G) = 0.20$$
$$P(C|W) = 0.775$$
$$P(C|G) = 0.65$$
$$P(G \cap W) = 0.06$$
$$P(G \cap W \cap C) = 0.01$$

$P(C \cap W) = P(C|W)P(W)$



$P(C \cap G) = P(C|G)P(G)$
$= 0.65 \cdot 0.20$
$= 0.13$

Fill out a Venn diagram with the probabilities of all possibilities.

Independence

Two events A and B are called *independent* if

$$\boxed{P(A \cap B) = P(A)P(B).}$$

Memorize

Equivalent definitions of independence

The following statements are equivalent:

- $P(A \cap B) = P(A)P(B)$
- $P(A) = P(A|B)$
- $P(B) = P(B|A)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \overset{\text{ind}}{=} \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A).$$

Also: If A, B independent, so are the pairs of events A, B^c and A^c, B and A^c, B^c.

If A, B are mutually exclusive it means $A \cap B = \emptyset$

A, B are indep. means $P(A \cap B) = P(A)P(B)$.

Exercise: Flip a coin twice and let

$$H_1 = \text{heads on first flip}$$
$$H_2 = \text{heads on second flip}$$

Find $P(H_1 \cap H_2)$ assuming that the flips are independent.

$$P(H_1 \cap H_2) = P(H_1)P(H_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Exercise: Let

- A = flat tire
- B = forgot spare tube

Suppose that $P(A) = 0.02$ and $P(B) = 0.10$ and $P(A \cap B) = 0.002$.

Are the events independent?

$P(A)P(B) = 0.002 = P(A \cap B)$

Yes indep. because $P(A)P(B) = P(A \cap B)$.

R_1 = 1st person responds

R_2 = 2nd person responds

$\vdots$ R_{10} = 10th person responds

Exercise: Send survey to 10 people. Let R_i = person i responds, $i = 1, \ldots, 10$. Assume independence with probability of response 0.20. Give

1. $P($ Everyone completes survey $)$
2. $P($ No one completes survey $)$
3. $P($ At least one person completes the survey $)$

① $P(R_1 \cap R_2 \cap R_3 \cap \cdots \cap R_{10}) = P(R_1) \cdot P(R_2) \cdot \ldots \cdot P(R_{10})$

$= (.2)(.2) \cdot \ldots \cdot (.2)$

$= (.2)^{10}.$

② $P(R_1^c \cap R_2^c \cap \cdots \cap R_{10}^c)$

$= P(R_1^c) \cdot \ldots \cdot P(R_{10}^c) = (.8)^{10}$

③ $1 - (.8)^{10}$

Exercise: From STAT 515 fa 2019:

- 40 students in class
- 10 students got an ‘A’ on final exam
- 12 students got an ‘A‘ hw average
- 23 students did not get an ‘A‘ on the final or an ‘A’ hw average.

If a student is drawn at random from the class, are the events ‘A’ on final and ‘A’ hw average independent?

Bayes' Rule (simplified)

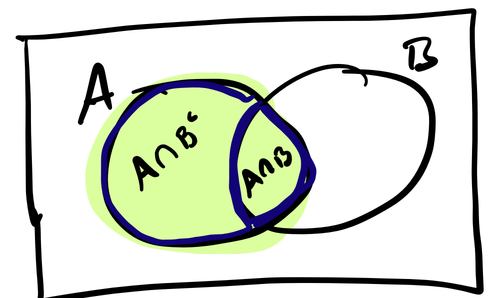
For any two events A and B such that $P(A) > 0$,

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}.$$

Useful when given $P(A)$, $P(B)$, $P(A|B)$,

but we really want

$P(B|A)$.



$P(A) = P(A \cap B^c) + P(A \cap B)$
$= P(A|B^c)P(B^c) + P(A|B)P(B)$

Show why... $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$

Exercise: Suppose 20% of a population has an infection. Suppose a test is positive with probability 0.70 when the infection is present and negative with probability 0.95 when the infection is absent.

For a randomly selected person from the population, find:

1. $P($ infection $|$ positive test $)$
2. $P($ no infection $|$ negative test $)$
3. If 100 people are tested, among whom 20 have the infection, how many do you expect of
   - False positives
   - True positives
   - False negatives
   - True negatives
4. Suppose a person is tested twice, with test outcomes independent. Find
   - $P($ infection $|$ two positive tests$)$
   - $P($ infection $|$ two negative tests$)$

Exercise: Suppose 20% of a population has an infection. Suppose a test is positive with probability 0.70 when the infection is present and negative with probability 0.95 when the infection is absent.

$+$ = positive test

$-$ = negative

I = infected

I^c = not infected

$P(I) = 0.20$

$P(+ \mid I) = 0.70$

$P(- \mid I^c) = 0.95$

What about $P(I \mid +)$? $P(I^c \mid -)$?

$$P(I \mid +) = \frac{P(+ \mid I)\,P(I)}{P(+ \mid I)\,P(I) + \underbrace{P(+ \mid I^c)}_{1 - P(- \mid I^c)}\,\underbrace{P(I^c)}_{1 - P(I)}}$$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}.$$

$$= \frac{(0.7)(0.20)}{(0.7)(0.20) + (0.05)(0.80)}$$

$$= 0.78$$

Exercise: Suppose 20% of a population has an infection. Suppose a test is positive with probability 0.70 when the infection is present and negative with probability 0.95 when the infection is absent.

$$\boxed{P(+ \mid I) = 0.70}$$ sensitivity

For a randomly selected person from the population, find:

$$P(- \mid I^c) = 0.95$$ specificity

1. $P($ infection $\mid$ positive test $)$ ✓
2. $P($ no infection $\mid$ negative test $)$ ←

$$P(I) = 0.20$$

$$P(I^c \mid -) = \frac{P(- \mid I^c)\,P(I^c)}{P(- \mid I^c)\,P(I^c) + \underbrace{P(- \mid I)}\,P(I)}$$

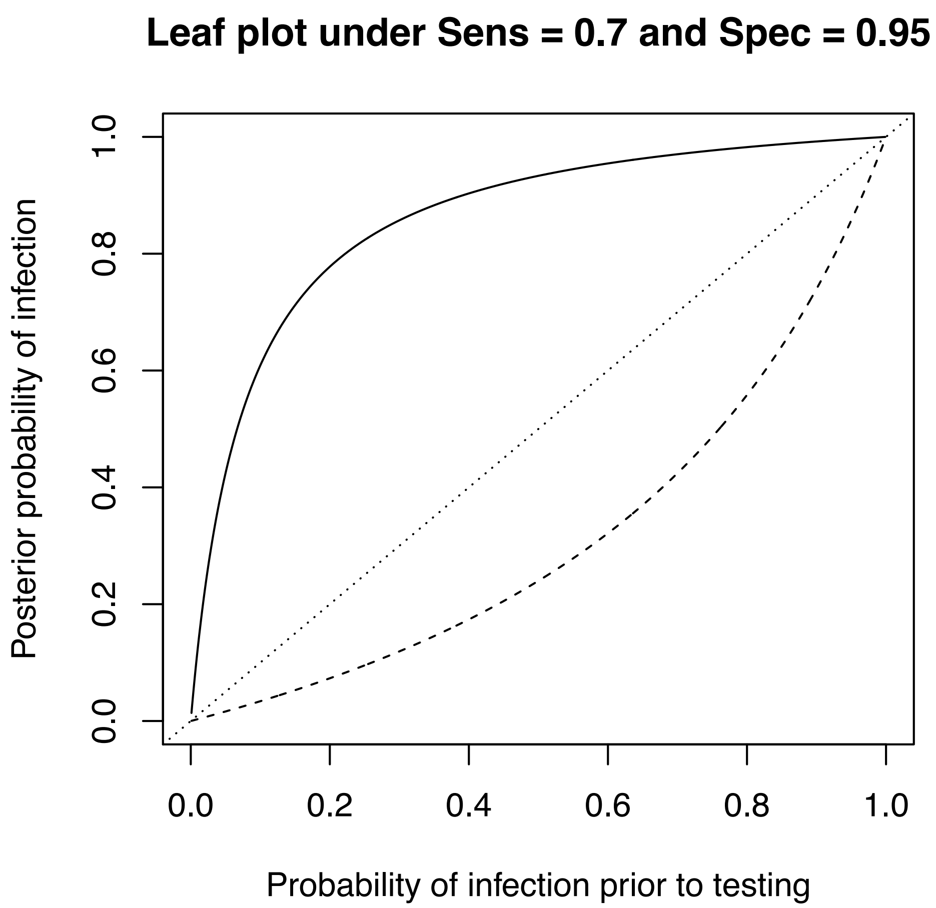
$I^c = B$

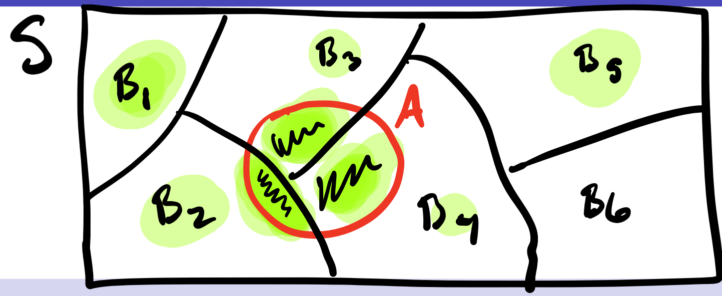
$$= \frac{0.95 \cdot (1 - 0.20)}{0.95\,(1 - 0.20) + (1 - 0.7)(0.20)}$$

$$= \frac{(.95)(.8)}{(.95)(.8) + (.3)(.2)}$$

$$= 0.927.$$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}.$$


Leaf plot under Sens = 0.7 and Spec = 0.95
Posterior probability of infection
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
Probability of infection prior to testing



$$P(B_2|A) = \frac{P(A\cap B_2)}{P(A)} = \frac{P(A|B_2)P(B_2)}{\underbrace{P(A\cap B_1)}_{P(A|B_1)P(B_1)} + P(A\cap B_2) + \dots + P(A\cap B_k)}$$

$=$

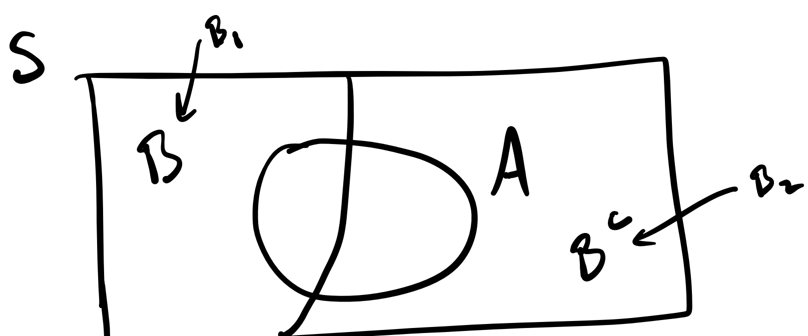
Bayes' Rule

For an event A and a set of mutually exclusive events $B_1, \ldots, B_K$ such that $P(B_1) + \cdots + P(B_K) = 1$, we have

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{P(A|B_1)P(B_1) + \cdots + P(A|B_K)P(B_K)}.$$

Show why...

Let $K = 2$ with $B_1 = B$ and $B_2 = B^c$ to get simplified version.



Exercise: Roll a die and draw one bill from a bag as follows:

Roll ⚀, ⚁, ⚂	$\longrightarrow$	draw from bag 1:	3 \$5 bills and 1 \$20 bill
Roll ⚃, ⚄	$\longrightarrow$	draw from bag 2:	2 \$5 bills and 2 \$20 bills
Roll ⚅	$\longrightarrow$	draw from bag 3:	1 \$5 bill and 3 \$20 bills

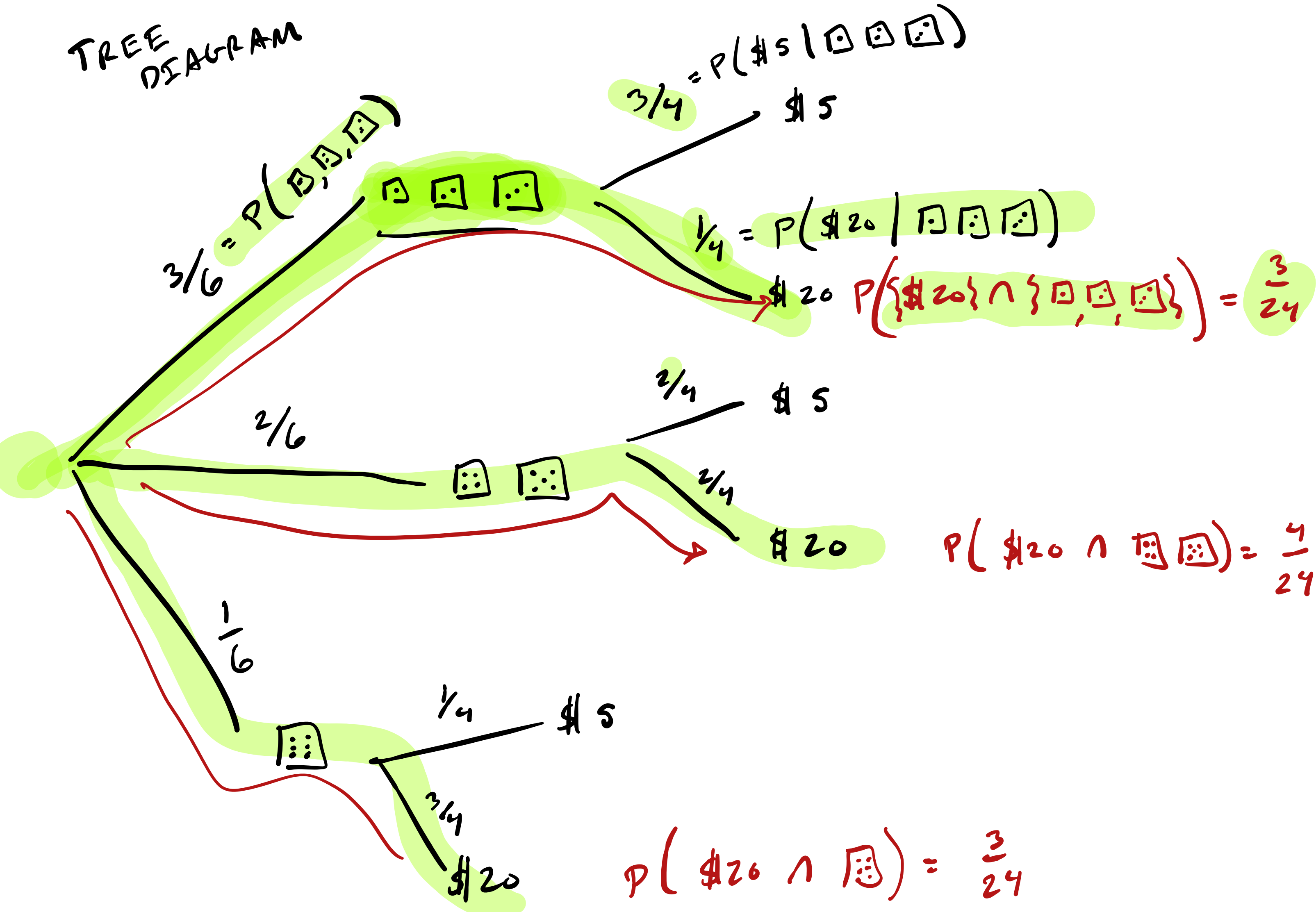
1. What is the probability that you get \$20?
2. Given that you get \$20, what is the probability that you $\boxed{\text{drew from bag 1?}}$
3. If you did this 1000 times:
   - How many times would you expect to get \$20?
   - Of the times you get \$20, on how many do you expect it to be from bag 1?

$$② \; P(B_1 \mid A) = \frac{P(A \cap B_1)}{P(A)} = \frac{P(A \mid B_1)\,P(B_1)}{P(A \mid B_1)\,P(B_1) + P(A \mid B_2)\,P(B_2) + P(A \mid B_3)\,P(B_3)}$$

$\uparrow$ \$20

$$= \frac{3/24}{10/24} = 3/10.$$

TREE DIAGRAM

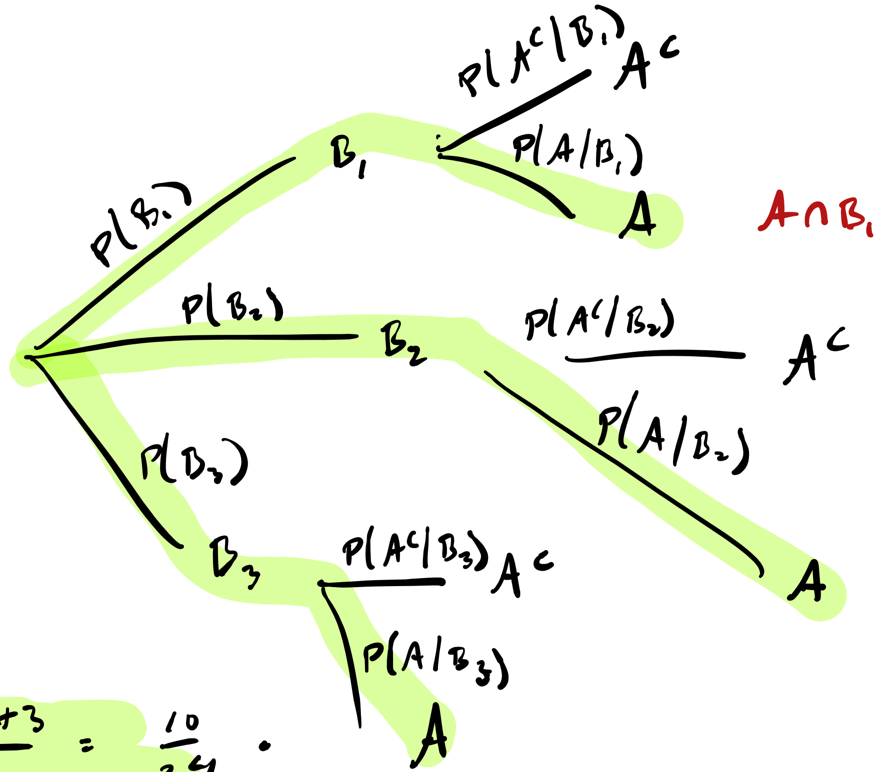


$A = \$20$

$B_1 = \{\ \ ,\ \ \}$

$B_2 = \{\ \ \}$

$B_3 = \{\ \}$



$$P(A) = \frac{3 + 4 + 3}{24} = \frac{10}{24}.$$