STAT 515 fa 2023 Lec 11 slides

Variance estimation

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

The chi-squared distributions

The probability distribution with pdf given by

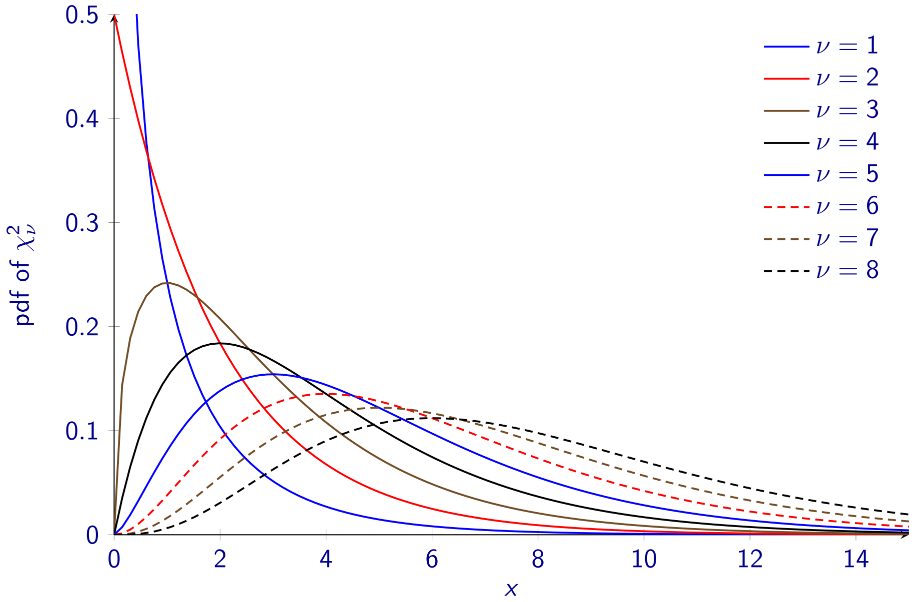
$$f(x) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}} x^{\nu/2-1} \exp\left(-\frac{x}{2}\right), \quad x > 0,$$

is called the *chi-squared distribution* with *degrees of freedom* $\nu > 0$.

For a rv W with this distribution, we write $W \sim \chi^2_\nu$.

How to build a χ^2_k random variable

If $W = Z_1^2 + \cdots + Z_k^2$, where $Z_1, \ldots, Z_k \overset{\text{ind}}{\sim} \text{Normal}(0,1)$, then $W \sim \chi^2_k$.


0.5
0.4
0.3
0.2
0.1
0
pdf of χ^2_ν
0
2
4
6
8
10
12
14
x
$\nu = 1$
$\nu = 2$
$\nu = 3$
$\nu = 4$
$\nu = 5$
$\nu = 6$
$\nu = 7$
$\nu = 8$

Recall: For a rs $X_1, \ldots, X_n$ the *sample variance* is the quantity

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

Sampling distribution of the sample variance

If $X_1, \ldots, X_n \stackrel{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, then

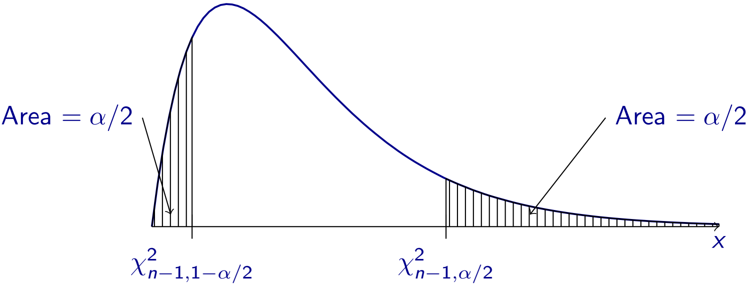
$$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}.$$

Exercise: Let $X_1, \dots, X_9 \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$ and use the fact that

$$P\left(2.18 < \frac{(9-1)S_9^2}{\sigma^2} < 17.53\right) = 0.95$$

to build a 95% confidence interval for σ^2.

What about a general $(1-\alpha)\times 100\%$ CI for any $\alpha \in (0,1)$?



Example: For $\alpha = 0.05$, $n = 9$, can get

$$\chi^2_{9-1,0.975} = \texttt{qchisq(0.025,9-1)} = 2.179731$$
$$\chi^2_{9-1,0.025} = \texttt{qchisq(0.975,9-1)} = 17.53455$$

Can also use `chi-square-table`.

Confidence interval for variance of a Normal population

Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$. Then

$$\left(\frac{(n-1)S_n^2}{\chi^2_{n-1,\alpha/2}}, \frac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha/2}} \right)$$

is a $(1-\alpha) \times 100\%$ confidence interval for σ^2.

Exercise: Run `data(Loblolly)` in R and consider the heights of 10-yr-old trees.

1. Make a Normal Q-Q plot of the data.
2. Build a 95% CI for the variance of the tree heights.
3. Build a 99% CI for the variance of the tree heights.