STAT 515 fa 2023 Lec 15 slides

p-values

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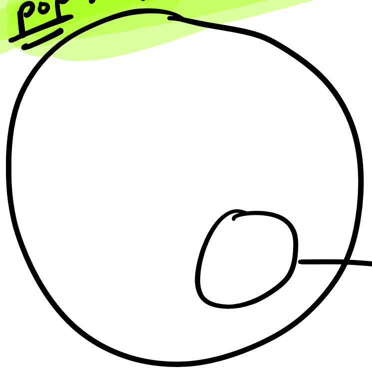
These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

pop: μ, σ^2, p



sample $\rightarrow$ $\bar{X}_n$, S_n^2, $\hat{p}_n$

$\bar{X}_n \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

$\bar{X}_n \pm t_{n-1,\alpha/2} \frac{S_n}{\sqrt{n}}$

$\hat{p}_n \pm z_{\alpha/2} \sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}$

Type I : H_0 true, but we reject it

Type II : H_0 false, we fail to reject.

H_0: statement about μ, or p (always contains "equals") $\leq$, $\geq$, $=$

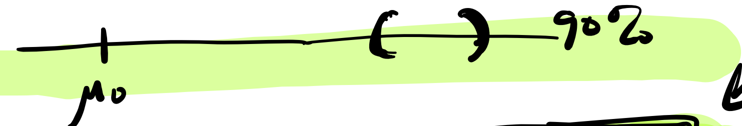
H_1: opposite statement about μ, or p $>$, $<$, $\neq$

Sig. level α is the prob. we make a Type I error.

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$

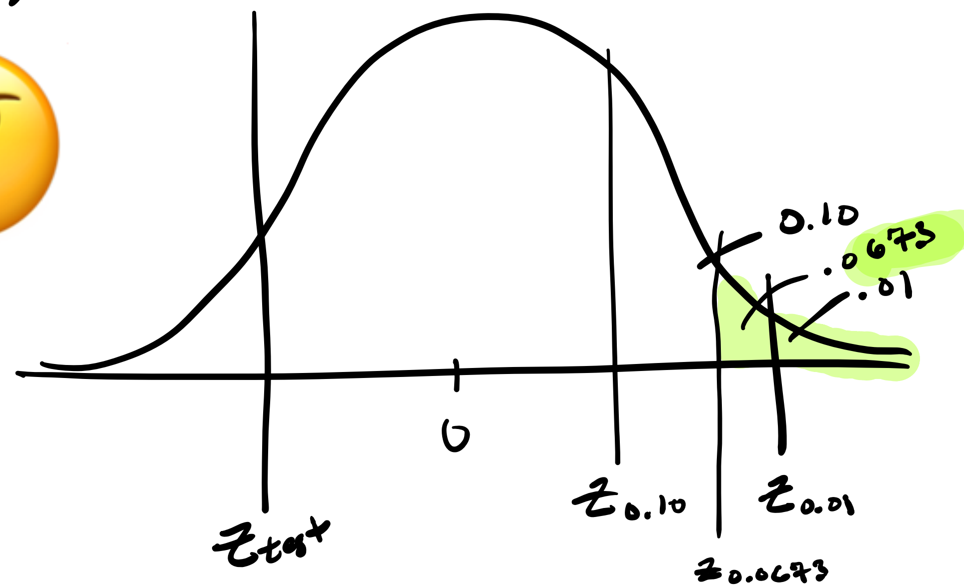
Discuss: Consider the case of Vinaya and her younger brother Anuj, who wish to test H_0 vs H_1. Each gathers data, and



- Anuj rejects H_0 based on a test with significance level $\boxed{\alpha = 0.10}$ and
- Vinaya rejects H_0 based on a test with significance level $\boxed{\alpha = 0.01}$.

Whose result is more "significant"?

Vinaya's result is more significant.



Anuj rejects if $Z_{test} > z_{0.10}$

Vinaya rejects if $Z_{test} > z_{0.01}$

At what significance levels would the observed data lead to a rejection of H_0?

This is a way to measure the strength of observed evidence against H_0.

The p-value

The smallest significance level α at which the observed data would lead to a rejection of H_0 is called the *p-value*.

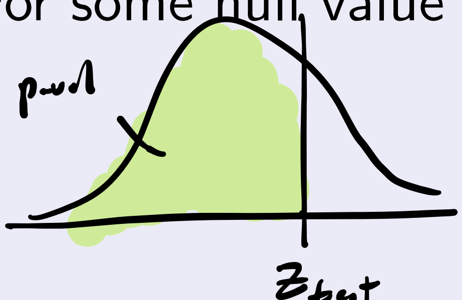
Interpretion: probability (under H_0) of observing data that carry as much or more evidence against the null as the data observed.

Once we have the p-value, we reject $\boxed{H_0}$ if p-value $< \alpha$.

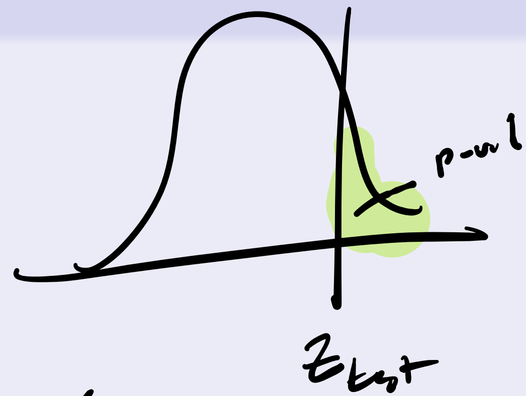
Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, σ^2 known.

Tests about μ when σ is known

For some null value μ_0, define the test statistic



$$Z_{\text{test}} = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}.$$



Then the following tests have $P(\text{Type I error}) \leq \alpha$.

$H_0\colon \mu \geq \mu_0$ $H_1\colon \mu < \mu_0$	$H_0\colon \mu = \mu_0$ $H_1\colon \mu \neq \mu_0$	$H_0\colon \mu \leq \mu_0$ $H_1\colon \mu > \mu_0$
Reject H_0 if $Z_{\text{test}} < -z_\alpha$	Reject H_0 if $\lvert Z_{\text{test}}\rvert > z_{\alpha/2}$	Reject H_0 if $Z_{\text{test}} > z_\alpha$
$p\text{-val} = P(Z < Z_{\text{test}})$	$p\text{-val} = 2 \cdot P(Z > \lvert Z_{\text{test}}\rvert)$	$\boxed{p\text{-val} = P(Z > Z_{\text{test}})}$

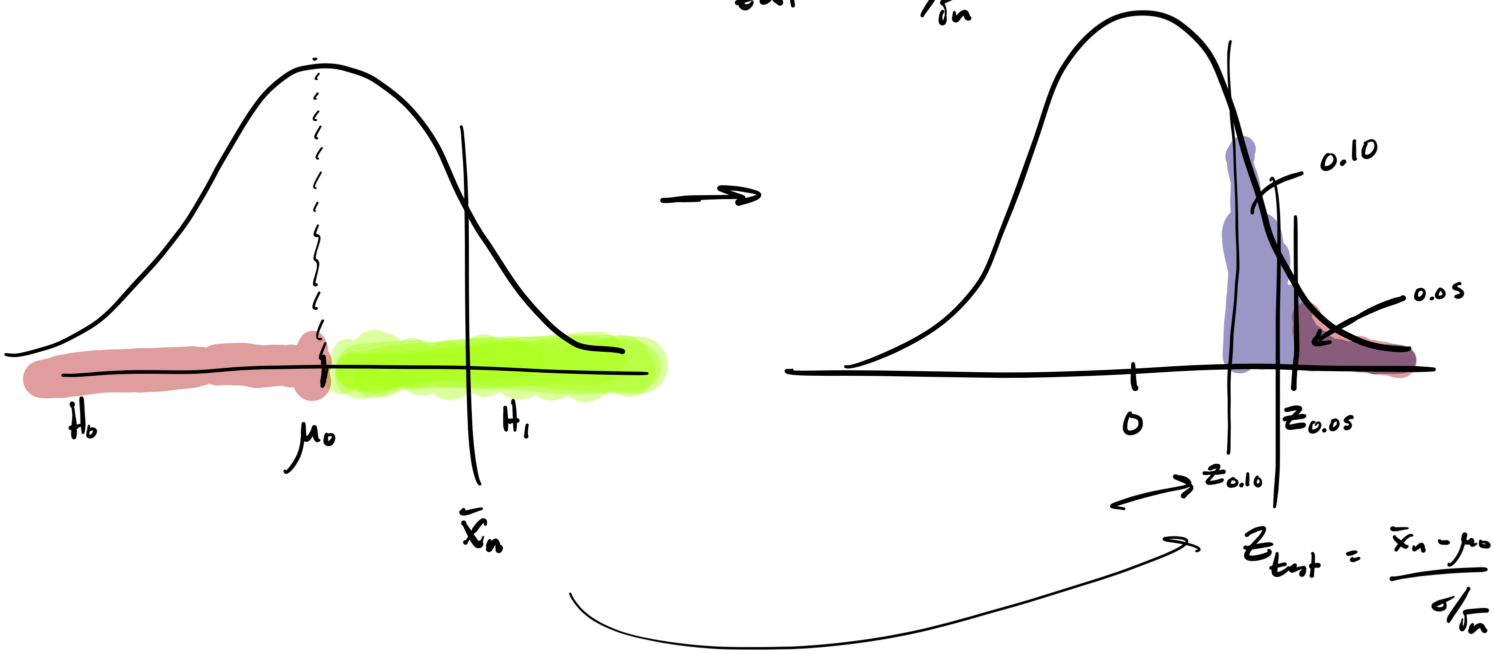
Discuss: Draw pictures of how to get the p-values.

data $\sim$ Normal(μ, σ^2), σ^2 know.

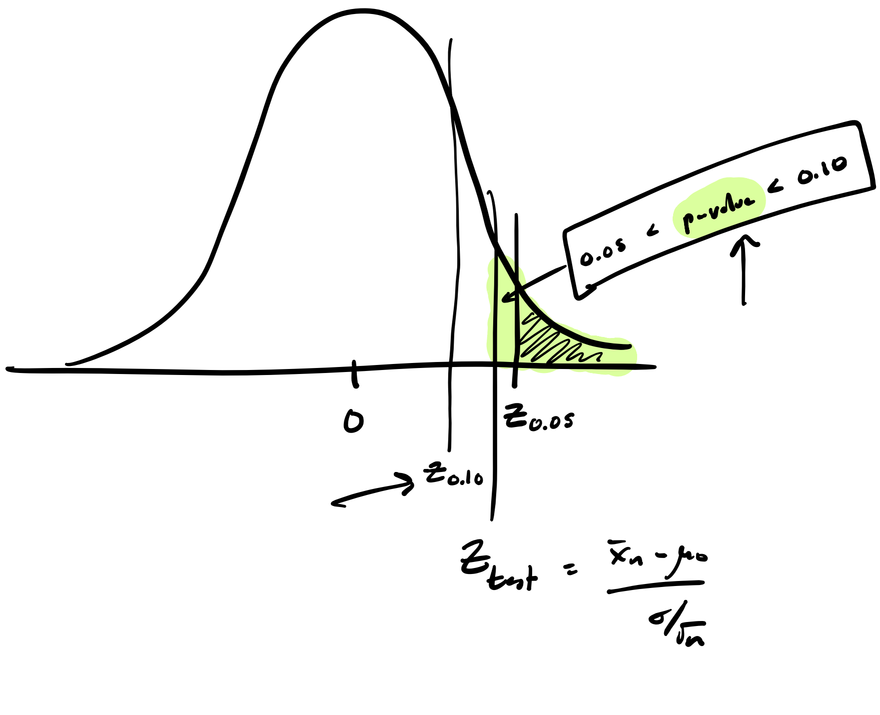
H_0: $\mu \leq \mu_0$

H_1: $\mu > \mu_0$

$$Z_{test} = \frac{\bar{x}_n - \mu_0}{\sigma/\sqrt{n}}$$

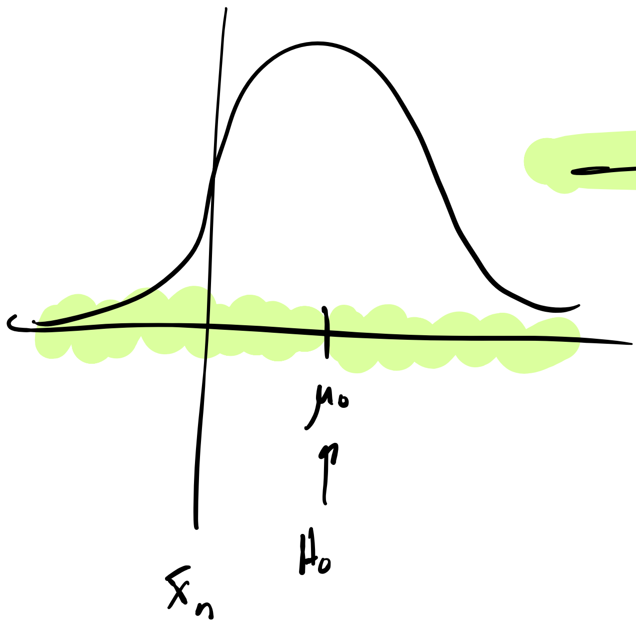


Reject H_0 if $Z_{test} > Z_\alpha$

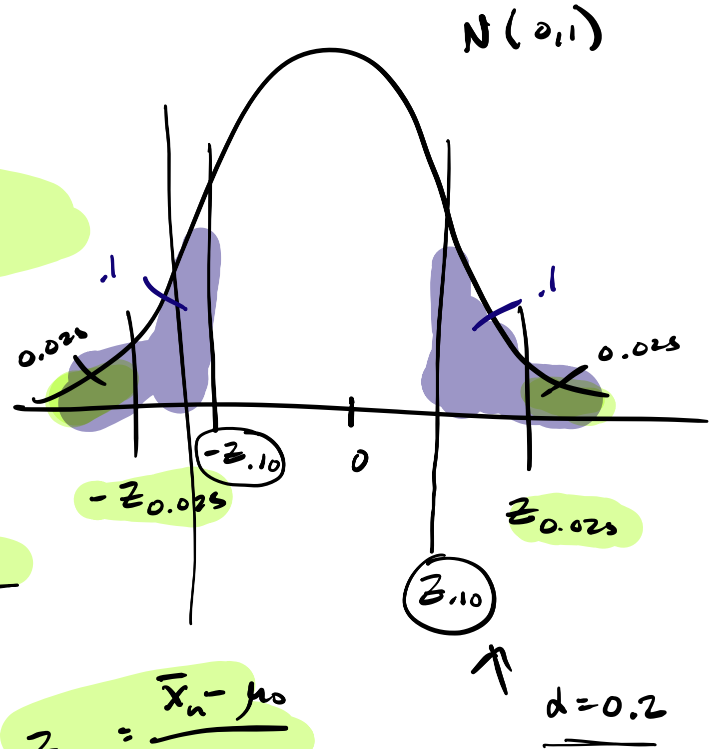


$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$

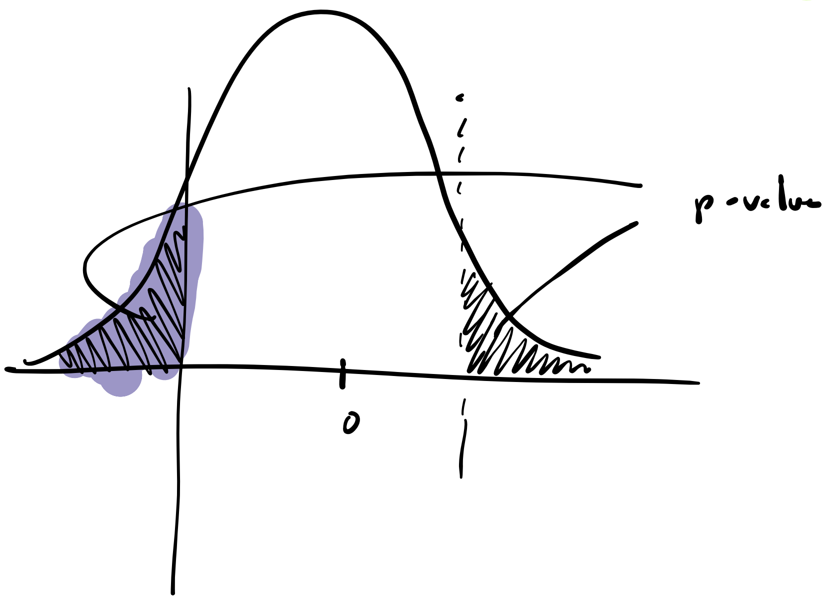


$$Z_{test} = \frac{\bar{x}_n - \mu_0}{\sigma/\sqrt{n}}$$



$\alpha = 0.05$

$$Z_{test} = \frac{\bar{x}_n - \mu_0}{\sigma/\sqrt{n}}$$



$$Z_{test} = \frac{\bar{x}_n - \mu_0}{\sigma/\sqrt{n}}$$

Exercise: Suppose a bottler of soft-drinks claims that its bottling process results in an internal pressure of 157 psi with standard deviation $\sigma = 3$ psi. You want to know whether the mean pressure is less than 157 (Ex 6.92 in [1]).

With $n = 40$ you get $\bar{X} = 155.7$. Find the p-value for testing

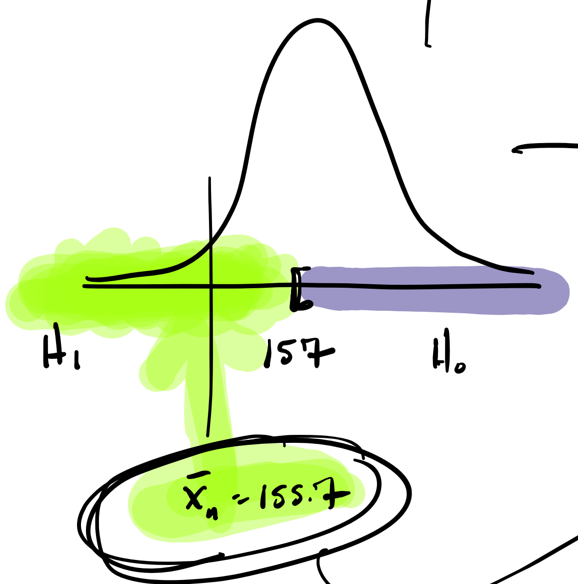
1. H_0: $\mu \geq 157$ vs H_1: $\mu < 157$
2. H_0: $\mu \leq 157$ vs H_1: $\mu > 157$
3. H_0: $\mu = 157$ vs H_1: $\mu \neq 157$

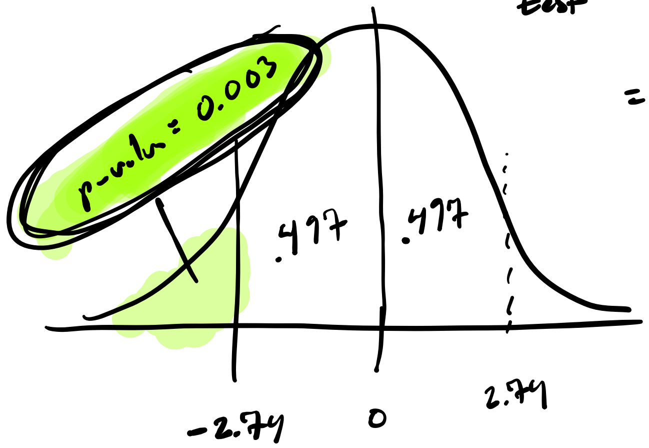
Assume the internal pressures are Normally distributed.

1. $H_0: \mu \geq 157 \leftarrow \mu_0$ $\quad n = 40 \quad \sigma = 3$

   $H_1: \mu < 157$ $\quad \bar{x}_n = 155.7$

$$z_{test} = \frac{\bar{x}_n - \mu_0}{\sigma/\sqrt{n}} = \frac{155.7 - 157}{3/\sqrt{40}} = -2.74$$



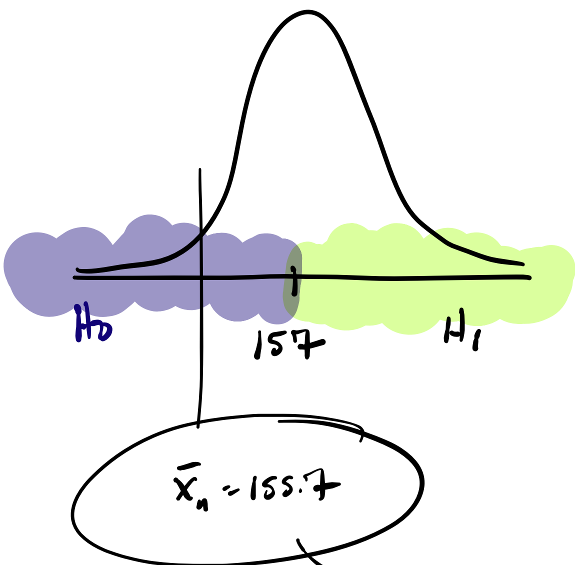


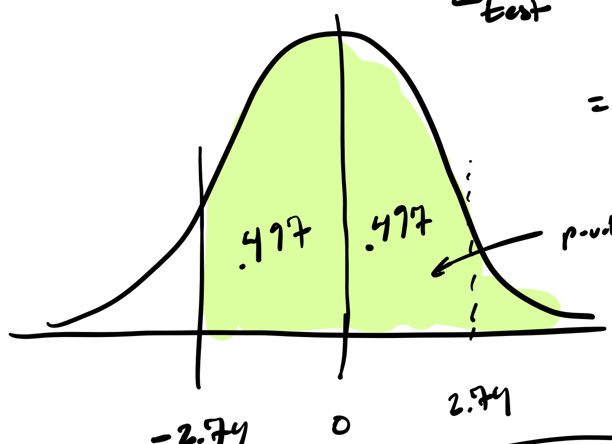
p-value is area under curve in the direction of the alternate hypothesis.

2. $H_0: \mu \leq 157$

   $H_1: \mu > 157$

$$z_{test} = \frac{\bar{x}_n - \mu_0}{\sigma/\sqrt{n}} = \frac{155.7 - 157}{3/\sqrt{40}} = -2.74$$





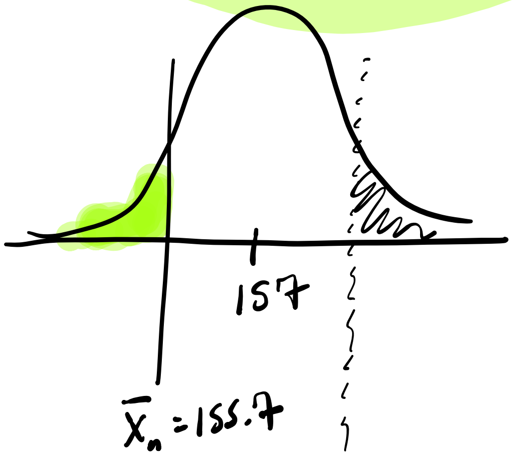
p-value = 0.987

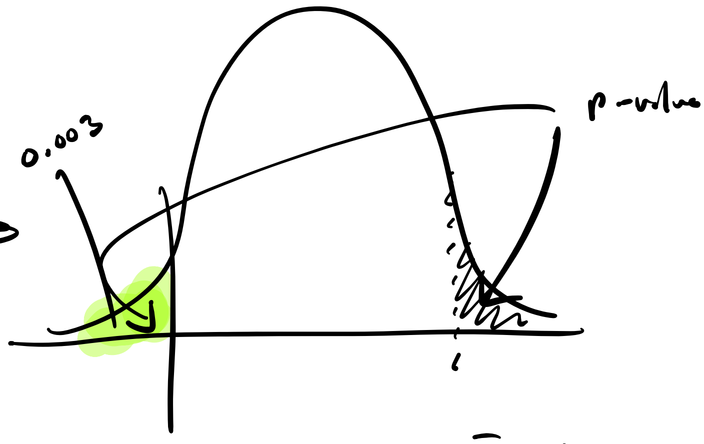
3

$H_0: \mu = 157$

$H_1: \mu \neq 157$

Add the area in the two tails.





$$-2.74 = z_{test} = \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}}$$

$$\boxed{\text{p-value} = 0.006.}$$

Do at home

Exercise: A machine should produce ball bearings with diameters having mean 0.5 in. and std. dev. $\sigma = 0.001$ in. Is the mean truly 0.5 in.? (Ex 6.84 in [1]).

With $n = 5$ you get $\bar{X}_n = 0.499$. Find the p-value for testing

1. H_0: $\mu \geq 0.5$ vs H_1: $\mu < 0.5$
2. H_0: $\mu \leq 0.5$ vs H_1: $\mu > 0.5$
3. H_0: $\mu = 0.5$ vs H_1: $\mu \neq 0.5$

Assume the ball bearing diameters are Normally distributed.

Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, σ^2 unknown.

Tests about μ when σ is unknown

For some null value μ_0, define the test statistic

$$T_{\text{test}} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}.$$

Then the following tests have $P(\text{Type I error}) \leq \alpha$.

$H_0\colon \mu \geq \mu_0$ $H_1\colon \mu < \mu_0$	$H_0\colon \mu = \mu_0$ $H_1\colon \mu \neq \mu_0$	$H_0\colon \mu \leq \mu_0$ $H_1\colon \mu > \mu_0$
Reject H_0 if $T_{\text{test}} < -t_{n-1,\alpha}$	Reject H_0 if $\lvert T_{\text{test}}\rvert > t_{n-1,\alpha/2}$	Reject H_0 if $T_{\text{test}} > t_{n-1,\alpha}$
$p\text{-val} = P(T < T_{\text{test}})$	$p\text{-val} = 2 \cdot P(T > \lvert T_{\text{test}}\rvert)$	$p\text{-val} = P(T > T_{\text{test}})$

For computing the p-values, let $T \sim t_{n-1}$. **Draw pictures.**

Exercise: Suppose you wish to test whether the LDL (bad cholesterol) level of South Carolinians exceeds the nationwide mean of 150 mg/dl.

With $n = 20$ you get $\bar{X}_n = 162.5$ and $S_n = 27.6$. Find the p-value for testing

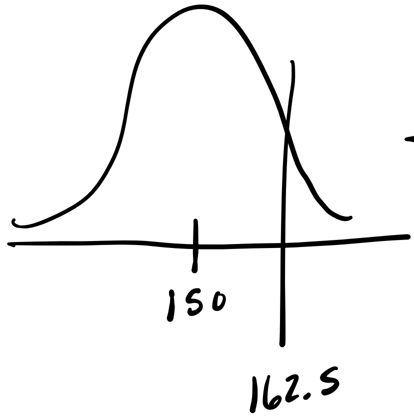
1. H_0: $\mu \geq 150$ vs H_1: $\mu < 150$
2. H_0: $\mu \leq 150$ vs H_1: $\mu > 150$
3. H_0: $\mu = 150$ vs H_1: $\mu \neq 150$

Assume the LDL levels are Normally distributed.

1 $H_0: \mu \geq 150$ vs $H_1: \mu < 150$ $\bar{X}_n = 162.5$, $S_n = 27.6$

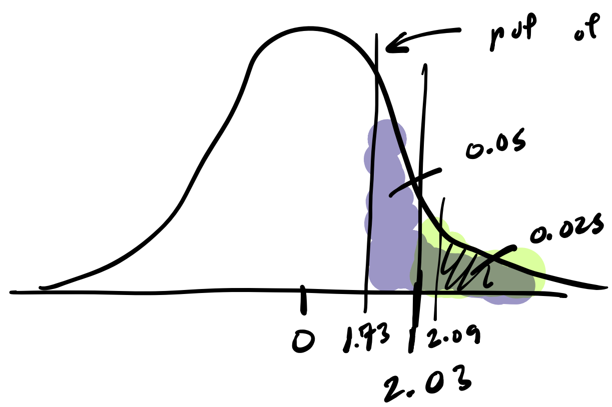
$n = 20$

2 $H_0: \mu \leq 150$ vs $H_1: \mu > 150$

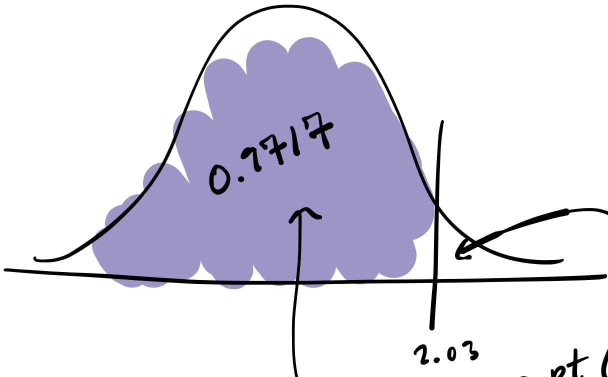


$$T_{test} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} = \frac{162.5 - 150}{27.6/\sqrt{20}} = 2.03$$

pdf of t_{20-1} dist



$0.025 < \text{p-value} < 0.05$



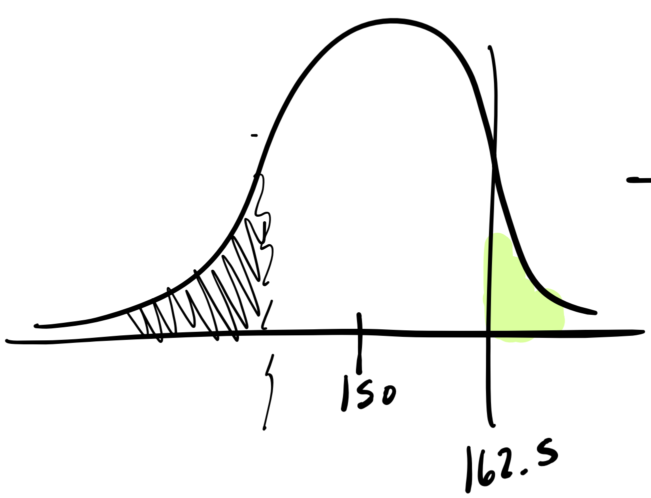
1 − pt(2.03, df = 19) = 0.0283.

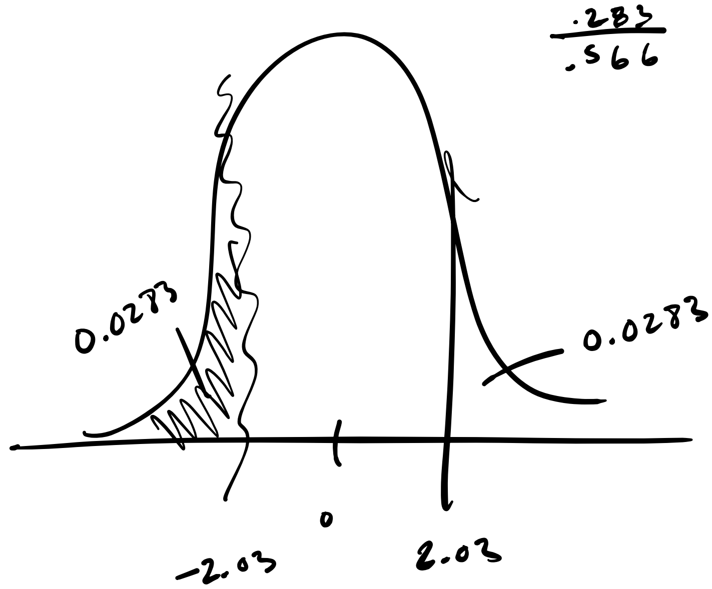
pt(2.03 , df = 19)

③ H_0: $\mu = 150$

H_1: $\mu \neq 150$

1
.273
.283
.566





p-value = 0.0566.

Let $X_1, \dots, X_n$ iid non-Normal with mean μ and unknown variance σ^2.

Large-n tests about μ when data are non-Normal

For some null value μ_0, define the test statistic

$$T_{\text{test}} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}.$$

Then for large n, the following tests have (approximately) $P(\text{Type I error}) \leq \alpha$.

$H_0\colon \mu \geq \mu_0$ $H_1\colon \mu < \mu_0$	$H_0\colon \mu = \mu_0$ $H_1\colon \mu \neq \mu_0$	$H_0\colon \mu \leq \mu_0$ $H_1\colon \mu > \mu_0$
Reject H_0 if $T_{\text{test}} < -z_\alpha$	Reject H_0 if $\lvert T_{\text{test}}\rvert > z_{\alpha/2}$	Reject H_0 if $T_{\text{test}} > z_\alpha$
$p\text{-val} = P(Z < T_{\text{test}})$	$p\text{-val} = 2 \cdot P(Z > \lvert T_{\text{test}}\rvert)$	$p\text{-val} = P(Z > T_{\text{test}})$

Exercise:

1. Draw a random sample of size $n = 35$ from the 2009 Boston Marathon `women's finishing times` and compute the p-value for testing

$$H_0\colon \mu \leq 4 \text{ versus } H_1\colon \mu > 4$$

2. Repeat this 1000 times and make a histogram of the p-values.

Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$.

Tests about p

For some null value p_0, define the test statistic

$$Z_{\text{test}} = \frac{\hat{p}_n - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}.$$

Then the following tests have (approximately) $P(\text{Type I error}) \leq \alpha$.

$H_0\colon p \geq p_0$ $H_1\colon p < p_0$	$H_0\colon p = p_0$ $H_1\colon p \neq p_0$	$H_0\colon p \leq p_0$ $H_1\colon p > p_0$
Reject H_0 if $Z_{\text{test}} < -z_\alpha$	Reject H_0 if $\lvert Z_{\text{test}} \rvert > z_{\alpha/2}$	Reject H_0 if $Z_{\text{test}} > z_\alpha$
$p\text{-val} = P(Z < Z_{\text{test}})$	$p\text{-val} = 2 \cdot P(Z > \lvert Z_{\text{test}} \rvert)$	$p\text{-val} = P(Z > Z_{\text{test}})$

Discuss: Draw pictures of how to get the p-values.

Exercise: The DNR will take action if an invasive fish is concluded to comprise more than 10% of the fish population in a habitat. In a random sample of 527 fish, 70 were of the invasive species.

1. What are the appropriate null and alternate hypotheses?
2. What is the p-value?
3. What would the p-value be if the two-sided test were of interest?

① $H_0: \; p \le 0.10$

$H_1: \; p > 0.10$

② $\hat{p}_n = \frac{70}{527} = 0.133$

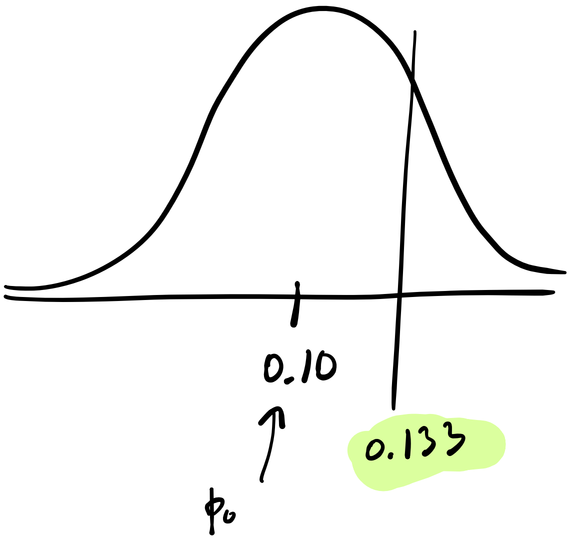
③ $H_0: \; p = 0.10$

$H_1: \; p \neq 0.10$

$p\text{-value} = 2(0.0057)$

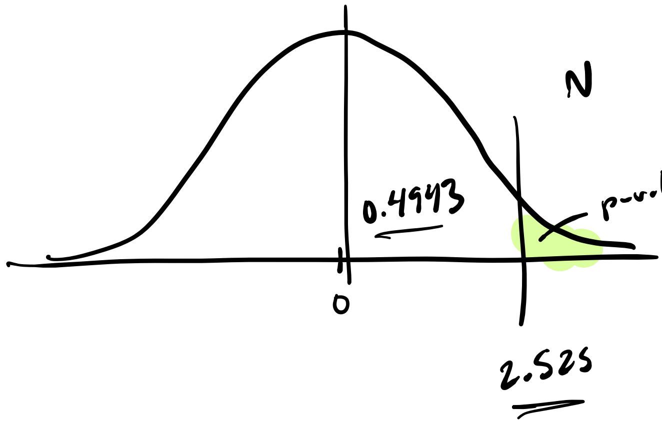
$= 0.0114.$

$$z_{test} = \frac{\hat{p}_n - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.133 - 0.10}{\sqrt{\frac{0.10(1-0.10)}{527}}} = 2.525$$



$\longrightarrow$

$N(0,1)$.



p-value = 0.0057.

Typical to use $\alpha = 0.05$

J.T. McClave and T.T. Sincich.
Statistics.
Pearson Education, 2016.