STAT 515 fa 2020 Exam I

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This is a take-home test due to COVID-19. Do not communicate with classmates about the exam until after its due date/time. You may

- *Use your notes and the lecture notes.*
- *Use books.*
- *NOT work together with others.*

Write all answers on blank sheets of paper; then take pictures and merge to a PDF. Upload a single PDF to Blackboard.

1. Copy down this sentence on your answer sheet and put your signature underneath: *I have not collaborated with any other student on this exam. The work I have presented is my own.*

2. Suppose you have 5 vacancies in your symphonic wind ensemble: 2 trombonists, 2 trumpeters, and 1 oboist are needed. There are 11 applicants for the positions: 5 trombonists, 4 trumpeters, and 2 oboists.

   (a) In how many ways could you select 2 from among the 5 trombonist applicants?

   We can do this in $\binom{5}{2} = 10$ ways.

   (b) In how many ways could the 5 vacancies be filled?

   We can fill the trombonist positions in $\binom{5}{2}$ ways, the trumpeter positions in $\binom{4}{2}$ ways, and the oboist position in $\binom{2}{1}$ ways, so the number of ways in which we can fill all the positions is

   $$\binom{5}{2}\binom{4}{2}\binom{2}{1} = 10 \cdot 6 \cdot 2 = 120.$$

   (c) You can conduct 3 auditions on Monday, 5 on Tuesday, and 3 on Wednesday. In how many ways can the applicants be assigned to the different days?

   This is a partition of the 11 applicants into 3 groups of sizes 3, 5, and 3, respectively. The number of ways to do this is

   $$\frac{11!}{3!5!3!} = 11 \cdot 5 \cdot 3 \cdot 8 \cdot 7 = 9{,}240.$$

   (d) Suppose you randomly select 3 of the applicants to be auditioned on Monday, and let X be the number of trombonists who are selected to be auditioned on Monday.

   i. What is the support of the random variable X?

   The random variable X can take the values in the set $\mathcal{X} = \{0, 1, 2, 3\}$.

   ii. What is the name of the probability distribution of X?

   It is the hypergeometric distribution. Specifically,

   $$X \sim \text{Hypergeometric}(N = 11, M = 5, K = 3).$$

   iii. Tabulate the probability distribution of X in a table like this:

x	$\ldots$
$P(X = x)$	$\ldots$

We have

$$P(X = x) = \frac{\binom{5}{x}\binom{11-5}{3-x}}{\binom{11}{3}}, \qquad \text{for } x = 0, 1, 2, 3.$$

So we get

x	0	1	2	3
$P(X = x)$	0.12121212	0.45454545	0.36363636	0.06060606
$P(X = x)$	4/33	15/33	12/33	2/33

iv. Give the expected value of X.

We have $\mathbb{E}X = 1.363636 = 15/11$.

3. Suppose you are a vendor of chocolate covered donuts—with and without sprinkles. Suppose 40% of your customers are kids and that kids ask for sprinkles 90% of the time, while adults ask for no sprinkles 80% of the time. *Hint: Draw a tree diagram for this one.*

(a) Find the probability that a randomly selected customer is a kid and asks for sprinkles.

Let K be the event that the customer is a kid and S that the customer asks for sprinkles. It is given that $P(S|K) = 0.90$, $P(S^c|K^c) = 0.80$ and $P(K) = 0.40$.

We have

$$P(K \cap S) = P(S|K)P(K) = 0.90 \cdot 0.40 = 0.36.$$

(b) Find the probability that a randomly selected customer asks for sprinkles.

We have (this can be seen from a tree diagram)

$$P(S) = P(S \cap K) + P(S \cap K^c) = P(S|K)P(K) + P(S|K^c)P(K^c) = 0.90 \cdot 0.40 + 0.20 \cdot 0.60 = 0.48.$$

(c) Find the probability that a randomly selected customer is a kid or asks for sprinkles.

We have

$$P(K \cup S) = P(K) + P(S) - P(K \cap S) = 0.40 + 0.48 - 0.36 = 0.52.$$

(d) Find the probability that a randomly selected customer asks for no sprinkles.

We have

$$P(S^c) = 1 - P(S) = 1 - 0.48 = 0.52.$$

(e) Find the probability that a randomly selected customer is an adult and asks for no sprinkles.

We have
$$P(S^c \cap K^c) = P(S^c|K^c)P(K^c) = 0.80 \cdot 0.60 = 0.48.$$

(f) Find the conditional probability that a customer is a kid given that the customer asks for sprinkles.

We have
$$P(K|S) = \frac{P(K \cap S)}{P(S)} = \frac{0.36}{0.48} = 3/4.$$

(g) Find the conditional probability that a customer is a not a kid given that the customer asks for no sprinkles.

We have
$$P(K^c|S^c) = \frac{P(K^c \cap S^c)}{P(S^c)} = \frac{P(S^c|K^c)P(K^c)}{0.80 \cdot 0.60} = \frac{0.48}{0.52} = 12/13 = 0.9230769.$$

4. In the game *Heckmeck am Bratwurmeck*, players begin each turn by rolling 8 dice. Each die is like an ordinary 6-sided die except that the "six" is replaced by the depiction of a smiling worm. Consider rolling the 8 dice:

(a) If X is the number of worms you roll, what is the name of the probability distribution of the random variable X?

The random variable X has the Binomial distribution. Specifically, $X \sim \text{Binomial}(8, 1/6)$.

(b) Give the probability of rolling no worms at all.

We have
$$P(X = 0) = (5/6)^8 = 0.232568.$$

(c) Give the probability of rolling at least one worm.

We have
$$P(X \geq 1) = 1 - P(X = 0) = 1 - 0.232568 = 0.767432.$$

(d) Give the probability that you roll exactly 4 worms.

We have

$$P(X=4) = \binom{8}{4}(1/6)^4(1-1/6)^{8-4} = \texttt{dbinom(4,8,1/6)} = 0.02604762.$$

(e) Give the probability that you roll 5 or more worms.

P(X $\geq$ 5) $= 1 - P(X \leq 4) = \texttt{1 - pbinom(4,8,1/6)} = 0.004608792.$

5. Let X be the body temperature of a randomly selected healthy person and assume X is Normally distributed with mean $\mu = 98$ and variance $\sigma^2 = 1/9$.

   (a) Find $P(X > 98)$.

   This is 1/2, since the pdf is symmetric around 98.

   (b) Give $P(X = 98)$.

   This is 0, since X is a continuous random variable.

   (c) Find $P(X > 98.5)$

   We have

$$P(X > 98.5) = P(Z > (98.5-98)/(1/3)) = P(Z > 1.5) = \texttt{1-pnorm(1.5)} = 0.0668072.$$

   (d) Find $P(97 < X < 99)$.

   We have

$$P(97 < X < 99) = P(-3 < Z < 3) = \texttt{pnorm(3) - pnorm(-3)} = 0.9973002.$$

   (e) Find the 40th percentile of X, that is the temperature such that X is less than or equal to it with probability 0.40.

   From `qnorm(.4)` $= -0.2533471$ we have $0.40 = P(Z < -0.2533471)$. So The 40th percentile of X is

$$98 + (1/3)(-0.2533471) = 97.91555.$$

   (f) Find the median of X, that is, the temperature such that X is less than or equal to it with probability 0.50.

The median is 98, since this is the value around which the pdf is symmetric.