STAT 515 fa 2021 Exam II

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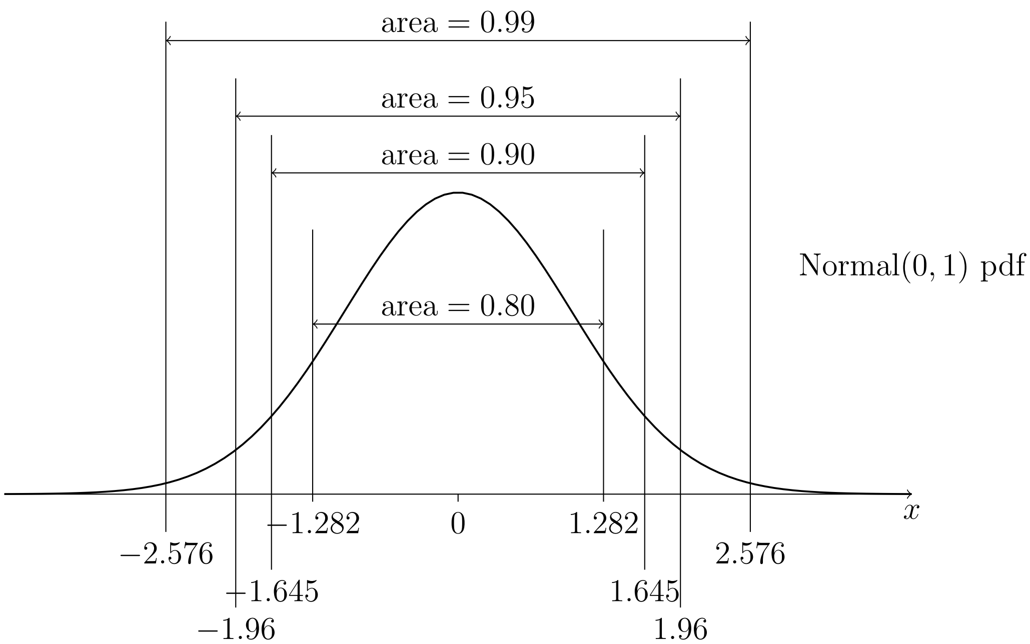
- Do not open this exam until told to do so.
- You may have two handwritten sheet of notes out during the exam.
- You have 75 minutes to work on this exam.
- You may use a simple calculator.
- If you are unsure of what a question is asking for, do not hesitate to ask me for clarification.
- *Good luck, and may the odds be ever in your favor!*



$$\hat{p}_n \pm z_{\alpha/2} \cdot \sqrt{\hat{p}_n(1-\hat{p}_n)/n} \qquad \bar{X}_n \pm t_{n-1,\alpha/2} \cdot S_n/\sqrt{n}$$

$$Z_{\text{test}} = \frac{\hat{p}_n - p_0}{\sqrt{p_0(1-p_0)/n}} \qquad T_{\text{test}} = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$

1. Contestants at a new carnival game win a prize if they toss a ring so that it lands around the neck of any of several rubber ducks floating in a whirlpool. The owner has set the cost of playing the game assuming no more than 3% of contestants will succeed. She will increase the cost if she finds that contestants succeed more often than this. In the first days of the carnival, she observes 729 contestants, of which 29 win. Then she comes to you for advice...

   (a) State the null and alternate hypotheses of interest to the owner of the game.

   The owner is interested in testing

   $$H_0\colon p \leq 0.03 \text{ versus } H_1\colon p > 0.03.$$

   (b) Give the value of the test statistic for testing the hypotheses in part (a).

   The test statistic is

   $$Z_{\text{test}} = \frac{29/729 - 0.03}{\sqrt{0.03(1-0.03)/729}} = 1.54803.$$

   (c) Do you reject the null hypothesis at the $\alpha = 0.05$ significance level?

   The test statistic value 1.54803 lies below the critical value $z_{0.05} = 1.645$ for testing the hypotheses. Therefore, we fail to reject the null hypothesis.

   (d) Interpret your statistical inference by giving a recommendation to the owner of the game.

   Since we have failed to reject the null hypothesis, we do not recommend that the owner increase the cost of playing the game.

   (e) Give an interval in which the p-value lies. Be as precise as you can.

   The p-value must be greater than 0.05, since the value of the test statistic was less than the $\alpha = 0.05$ critical value; however, the p-value cannot be greater than 0.10, since $z_{0.10} = 1.282$ (from the diagram on the front of the exam), which is less than the value of the test statistic. So the p-value lies in the interval $(0.05, 0.10)$.

   (f) Construct a 95% confidence interval for the proportion of contestants who win the game.

   The Wald-type 95% confidence interval is given by

   $$29/729 \pm 1.96\sqrt{29/729(1-29/729)/729} = (0.026, 0.054),$$

   and the Agresti-Coull interval is given by

   $$31/733 \pm 1.96\sqrt{31/733(1-31/733)/733} = (0.028, 0.057).$$

(g) If the owner is interested in the hypotheses

$$H_0\colon\ p = 0.03 \text{ versus } H_1\colon\ p \neq 0.03,$$

does she reject H_0 at $\alpha = 0.05$?

She does not reject $H_0\colon\ p = 0.03$. The critical value for the two-sided test is $z_{0.05/2} = 1.96$. Since the test statistic does not exceed this value, we would fail to reject the null hypothesis. In addition (though confidence intervals and tests of two-sided hypotheses are not exactly equivalent in the case of the proportion), the 95% confidence interval for the true proportion contains the value 0.03. We therefore do not reject $p = 0.03$ at the $\alpha = 0.05$ significance level.

(h) If the owner wishes to estimate the true proportion of contestants who win the game with a margin of error no greater than 1%, with 95% confidence, how many contestants should she observe? Use the data collected in the first days of the carnival as a best guess for p.

We should choose the smallest sample size n such that $1.96\sqrt{29/729(1 - 29/729)/n} \leq 0.01$. After doing some algebra, we see that we need

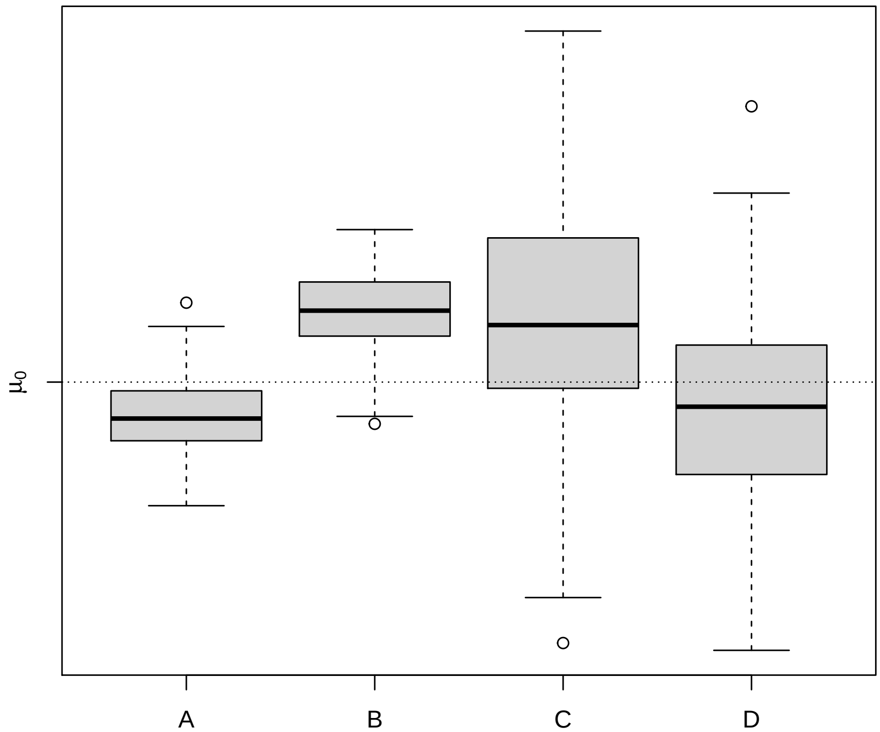
$$n \geq (1.96)^2 \cdot 29/729(1 - 29/729)/(0.01)^2 = 1467.416,$$

so we recommend $n = 1{,}468$.

2. Each of three scientists is interested in a set of hypotheses about a mean μ:

   1. Alain is interested in testing H_0: $\mu \leq \mu_0$ versus H_1: $\mu > \mu_0$.
   2. Muriel is interested in testing H_0: $\mu \geq \mu_0$ versus H_1: $\mu < \mu_0$.
   3. Étienne is interested in testing H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$.

   Consider these boxplots of data sets A, B, C, and D that the scientists might observe:



   Fill in each blank with A, B, C, or D.

   (a) For Alain, data set ___**A**___ would give the largest p-value, while data set ___**B**___ would give the smallest p-value.

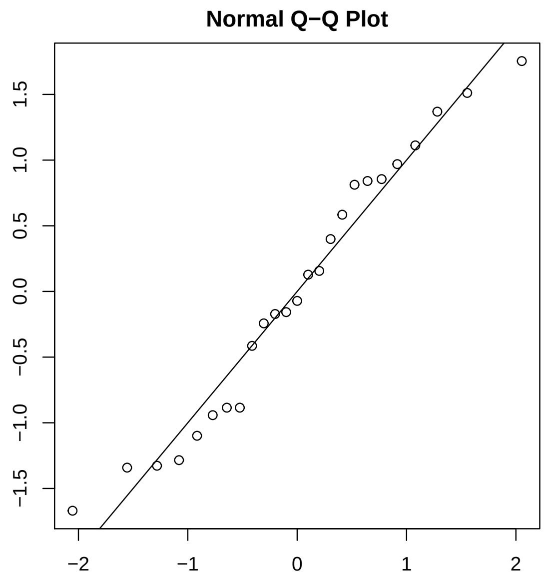
   (b) For Muriel, data set ___**B**___ would give the largest p-value, while data set ___**A**___ would give the smallest p-value.

   (c) For Étienne, data set ___**D**___ would give the largest p-value, while data set ___**B**___ would give the smallest p-value.

3. Each of a large number of barbunya beans was photographed by a computer and its size measured in the number of pixels covered by the bean [1]. Here is a sample of 25 of these measurements, in thousands of pixels, rounded to the nearest decimal place:

71.5	69.5	76.5	67.7	70.1	73.4	76.3	64.4	71.7	62.9	58.9	80.2	
81.2	78.4	68.9	76.6	61.3	82.9	69.4	64.4	61.2	64.0	61.6	74.7	77.4

Suppose it is of interest to test whether the mean pixel area of barbunya beans is less than sixty-seven thousand pixels based on these 25 measurements. The sample mean and standard deviation for these data are $\bar{X}_n = 70.60$ and $S_n = 7.01$. A Normal quantile-quantile plot of the standardized values is here:



(a) State the null and alternate hypotheses of interest.

Letting μ represent the mean area in thousands of pixels of barbunya beans, it is of interest to test

$$H_0\colon\ \mu \geq 67 \text{ versus } H_1\colon\ \mu < 67.$$

(b) Explain why it is important to look at the Normal quantile-quantile plot.

The Normal quantile-quantile plot tells us whether we can assume the data were drawn from a population with Normally distributed values. We need this to be the case in order to conduct our t-distribution-based tests of hypotheses when the sample size is small (less than 30, say), as it is in this case. The Normal quantile-quantile plot of these data suggests that the data do come from a distribution with Normally distributed values.

(c) Give the value of the test statistic for testing the hypotheses in part (a).

The value of the test statistic is

$$T_{\text{test}} = \frac{70.60 - 67}{7.01/\sqrt{25}} = 2.57.$$

(d) Do you reject the null hypothesis at significance level $\alpha = 0.05$?

Since the value of the sample mean is $\bar{X}_n = 70.60$, the data support the null hypothesis. This means that we will fail to reject. We would reject the null hypothesis if our test statistic had a value less than $-t_{25-1,0.05} = -1.7109$ (obtained from the t-table). Since our test statistic takes a positive value, we fail to reject H_0.

(e) Give an interval in which the p-value lies. Be as precise as you can.

From the t-table, we see that the $\alpha = 0.01$ critical value is $-t_{25-1,0.01} = -2.4922$ and the $\alpha = 0.005$ critical value is $-t_{25-1,0.005} = -2.7969$. Since the test statistic value 2.57 lies between 2.4922 and 2.7969, the p-value must be in the interval $(0.990, 0.995)$. This can be seen most easily by drawing a picture and remembering to shade under the pdf of the t_{25-1} distribution in the direction of the alternate hypothesis.

(f) Construct a 95% confidence interval for the mean pixel area of barbunya beans.

A 95% confidence interval for the mean pixel area of barbunya beans is

$$70.60 \pm \underbrace{t_{25-1,0.025}}_{=2.0639} \cdot 7.01/\sqrt{25} = (67.71, 73.50).$$

(g) Based on these data, would you reject the null hypothesis when testing

$$H_0\colon \mu = 67 \text{ versus } H_1\colon \mu \neq 67$$

at the $\alpha = 0.05$ significance level?

Since the 95% confidence interval for μ does not contain the value 67, we would reject the null hypothesis $H_0\colon \mu = 67$ at the $\alpha = 0.05$ significance level.

References

[1] Koklu M. and Ozkan I.A. Multiclass classification of dry beans using computer vision and machine learning techniques. *Computers and Electronics in Agriculture*, 174(105507), 2020.