STAT 515 fa 2023 Exam I

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- Do not open this exam until told to do so.
- You may have one handwritten sheet of notes out during the exam.
- You have 75 minutes to work on this exam.
- You may NOT use any kind of calculator.
- If you are unsure of what a question is asking for, do not hesitate to ask me for clarification.
- *Good luck, and may the odds be ever in your favor!*

$X \sim$		$\mathcal{X}$	$\mathbb{E}X$	$\text{Var}(X)$
Binomial(n,p)	$P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$	$x=0,1,\dots,n$	np	$np(1-p)$
Poisson(λ)	$P(X=x)=\dfrac{\lambda^x e^{-\lambda}}{x!}$	$x=0,1,2,\dots$	λ	λ
Exponential(λ)	$P(X\leq x)=1-e^{-x\lambda}$	$x>0$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$

Lecture 5, slide 6: The # of free throws example is exactly like this question

1. Assume the number of broken eggs in a carton of 18 eggs follows a binomial distribution, with the probability of any egg being broken equal to 1/50.

   (a) Give the probability that no eggs in a carton are broken.

   Letting X be the number of broken eggs in a cartion, we have $X \sim \text{Binomial}(18, 1/50)$. So
   $$P(X = 0) = \binom{18}{0}(1/50)^0(1 - 1/50)^{18} = (49/50)^{18}.$$

   (b) Give the probability that at least one egg in a carton is broken.

   This is $P(X \geq 1) = 1 - P(X = 0) = 1 - (49/50)^{18}$.

   (c) Give an expression for the probability that exactly 7 eggs in a carton are broken.

   We have $P(X = 9) = \binom{18}{7}(1/50)^7(1 - 1/50)^{18-7}$.

   (d) If you bought 1,000 cartons of eggs, recorded the number of broken eggs in each one, and then averaged these 1,000 numbers, to what value do you expect your average would be close?

   The expected value of X is $18(1/50) = 18/50 = 0.36$.

Lecture 8, slide 10: the tire example is like this question

2. Suppose the number of cyclists passing by an observation point on a road in one hour is a Poisson random variable such that on average 4 cyclists pass by in an hour.

   (a) Give the probability that exactly 3 cyclists pass by in an hour (you do not have to simplify our expression to a number).

   Let $X \sim \text{Poisson}(\lambda = 3)$. Then $P(X = 3) = e^{-4}(3)^4/3!$.

   (b) Give an expression for the probability that at least one cyclist passes by.

   This is $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-4}(4)^0/0! = 1 - e^{-4}$.

   (c) Give the average time elapsed between the passing by of cyclists.

   The time elapsed between passing cyclists follows the $\text{Exponential}(\lambda = 4)$ distribution, which has mean $1/\lambda = 1/4$. So on average, the elapsed time between the passing of cyclists is a quarter of an hour.

   (d) Give the distribution of the number of cyclists passing by in two hours.

   This the $\text{Poisson}(\lambda = 8)$ distribution.

Lecture 3, slide 12: the example on the slide is exactly like this question

3. Two white wines and three red wines are to be blindly tasted. Your friend who considers himself a connoisseur will detest any white wine with probability 3/4 and detest any red wine with probability 1/4. He blindly selects a wine and tastes it...

   (a) Give the probability that he detests the randomly selected wine.

   Let D be the event that he detests the wine and R be event that he selects a red wine. We have

$$\begin{aligned} P(D) &= P(D \cap R) + P(D \cap R^c) \\ &= P(D|R)P(R) + P(D|R^c)P(R^c) \\ &= (1/4)(3/5) + (3/4)(2/5) \\ &= 9/20. \end{aligned}$$

   Drawing a tree is perhaps the best way to get the answer.

   (b) Suppose he detests the wine; with what probability has he selected a white wine?

   We have

$$P(R^c|D) = \frac{P(D \cap R^c)}{P(D)} = \frac{P(D|R^c)P(R^c)}{P(D)} = \frac{(3/4)(2/5)}{9/20} = 2/3.$$

   (c) Suppose he likes the wine; with what probability has he selected a white wine?

   We have

$$P(R^c|D^c) = \frac{P(D^c \cap R^c)}{P(D^c)} = \frac{P(D^c|R^c)P(R^c)}{P(D^c)} = \frac{(1/4)(2/5)}{11/20} = 2/11.$$

   (d) Are the events that he selects a red wine and that he detests the selected wine independent?

   They are not independent, because $P(D|R) = 1/4$ and $P(D) = 9/20$, so that $P(D|R) \neq P(D)$.

Lecture 7, slide 9: (a), (b), (c), and (d) of this question are exactly like the exercise on the slide

4. A European cookie recipe calls for 120 grams of flour. Suppose that when 1 cup of flour is measured out, the weight of the measured flour in grams follows a normal distribution with mean 130 and standard deviation 5. Lacking a kitchen scale, you measure the flour with your measuring cup and bake...

   (a) With what probability will you measure out too much flour?

   Let X be the amount of measured flour. We have

   $$P(X > 120) = P\left(Z > \frac{120 - 130}{5}\right) = P(Z > -2) = 0.5000 + 0.4772 = 0.9772,$$

   where we have used the Z-table.

   (b) With what probability will you measure out too little flour?

   This is $P(X < 120) = 1 - P(X < 120) = 1 - 0.9772 = 0.0228$.

   (c) With what probability will you measure out exactly 120 grams of flour?

   Since X is a continuous random variable, we have $P(X = 120) = 0$.

   (d) The cookies will turn out splendidly as long as the amount of flour used is between 110 and 140 grams. With what probability will the cookies turn out splendidly?

   We have

   $$\begin{aligned} P(110 < X < 140) &= P((110 - 130)/5 < Z < (140 - 130)/5) \\ &= P(-4 < Z < 2) \\ &= 0.5 + 0.4772 \\ &= 0.9772, \end{aligned}$$

   where we have used the Z-table.

   (e) Give the amount of flour in grams such that only 10% of measured cups of flour contain a greater amount.

   This is the 90th percentile of the distribution of X. The 90th percentile of the standard normal distribution is 1.28, which one can find on the Z-table. The 90th percentile of the distribution of X is found as $130 + 1.28(5) = 136.4$.

Lecture 7, slide 11: (e) of this question is exactly like part (1) of the exercise on the slide

Lecture 5, slide 9: the vaping example on the slide is exactly like this question

5. You may reach into a bag of candies and grab 3. The bag contains 5 candies, 2 of which are of your favorite kind. Let X be the number of your favorite kind of candy you grab.

   (a) Tabulate the probability distribution of X. Simplify all expressions.

The table is

x	0	1	2
$P(X = x)$	$\frac{\binom{2}{0}\binom{3}{3}}{\binom{5}{3}}$	$\frac{\binom{2}{1}\binom{3}{2}}{\binom{5}{3}}$	$\frac{\binom{2}{2}\binom{3}{1}}{\binom{5}{3}}$
	$1/10$	$6/10$	$3/10$

   (b) Give the expected value of X.

The expected value of X is $\mathbb{E}X = 0(1/10) + 1(6/10) + 2(3/10) = 12/10 = 6/5 = 1.2$.

   (c) Now suppose that the bag contains a practically infinite number of candies, with two-fifths of the candies of your favorite kind. You may reach in and grab 3. Tabulate the distribution of X, the number of your favorite kind of candy you grab. Simplify all expressions.

Now we have $X \sim \text{Binomial}(n = 3, p = 2/5)$, and the distribution is

x	0	1	2	3
$P(X = x)$	$\binom{3}{0}(2/5)^0(3/5)^{3-1}$	$\binom{3}{1}(2/5)^1(3/5)^{3-1}$	$\binom{3}{2}(2/5)^2(3/5)^{3-2}$	$\binom{3}{3}(2/5)^3(3/5)^{3-3}$
	$27/125$	$54/125$	$36/125$	$8/125$