STAT 515 hw 2

Counting, conditional probability, Bayes' rule, independence

1. An iteration of the board game Candyland has the following cards: 6 each of red, purple, yellow, blue, orange, and green cards, and then 4 each of double red, double purple, double yellow, double blue, double orange, and double green cards. In addition, there is one lollipop, one peppermint, one peanut, and one ice cream card.

   (a) If you draw 5 cards, one at a time, placing each card back into the deck and shuffling before the next draw, how many sequences of draws are possible?

   > There are 16 different cards, so there are $16^5 = 1{,}048{,}576$ different sequences of 5 draws, with replacement.

   (b) If you draw 5 cards, one at a time, without placing each card back before drawing the next, what is the probability that one of your 5 cards will be the ice cream cone?

   > We first count the number of 5 card "hands" with the ice cream cone card. The first task is to draw the ice cream cone. There is 1 way to do this. The second task is to draw the remaining 4 cards from among the remaining 63 cards. This can be done in $\binom{63}{4}$ ways. The total number of 5 card "hands" is $\binom{64}{5}$. So we have
   >
   > $$P(\text{get ice cream cone}) = \frac{1 \cdot \binom{63}{4}}{\binom{64}{5}} = \frac{5}{64}.$$

2. Three guests to a tea party sit down in three chairs. Then the host re-arranges them according to a pre-made seating chart

   (a) What is the probability that all of the guests may stay in their chairs?

   > There are $3! = 6$ possible seating arrangements. The guests will choose the correct one themselves with probability $1/6$.

   (b) What is the probability that at least one guest may stay in her chair? *Hint: Just write down all the sample points. Don't try to use counting rules.*

   > If we symbolize the host's seating arrangement as 123 and the possible arrangements the guests may choose as
   >
   > $$\mathcal{S} = \left\{ \begin{array}{ccc} 123 & 213 & 312 \\ 132 & 231 & 321 \end{array} \right\},$$
   >
   > we see that in 4 out of the 6 possible arrangements, at least one guest is seated in the right place, so the probability is $2/3$.

3. A train to the western frontier will consist of 4 passenger cars, 3 cattle cars, and 2 luggage cars, which are to be put in order at random.

   (a) Bandits plan to mount the train and enter the two rearmost cars. Find the probability that they enter

      i. two passenger cars.

      Choose 2 of the 4 passenger cars to put rearmost in $\binom{4}{2}$ ways; put them in order in 2! ways; then put the remaining 7 cars in order in 7! ways. There are 9! ways of ordering all the cars, so we have

      $$P(\text{two passenger cars}) = \frac{\binom{4}{2} \cdot 2! \cdot 7!}{9!} = \frac{1}{6}.$$

      We can also look at it like this: There are $\binom{9}{2}$ ways to select the two rearmost cars. There are $\binom{4}{2}$ ways to choose two passenger cars to be rearmost. So the answer is

      $$P(\text{two passenger cars}) = \frac{\binom{4}{2}}{\binom{9}{2}} = \frac{1}{6}.$$

      ii. a luggage car and a passenger car.

      We have

      $$P(\text{a luggage car and a passenger car}) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{9}{2}} = \frac{2}{9}.$$

      iii. at least one cattle car.

      We have

      $$\begin{aligned} P(\text{at least one cattle car}) &= P(\text{one cattle car}) + P(\text{two cattle cars}) \\ &= \frac{\binom{3}{1}\binom{6}{1}}{\binom{9}{2}} + \frac{\binom{3}{2}}{\binom{9}{2}} \\ &= \frac{1}{12} + \frac{1}{2} \\ &= \frac{7}{12}. \end{aligned}$$

   (b) A lady's shawl flies out a window of the foremost passenger car, reenters a window in the rearmost passenger car, and is seized by a gentleman who gallantly vows to return it while the train is in motion. With what probability can he make his way from the rearmost passenger car to the foremost, passing only through passenger cars?

First, line up all the passenger cars together; there are 4! ways of doing this. Now, treating the four passenger cars together as a single car (since we are not going to separate them), we have 6 "cars" to arrange. This can be done in 6! ways. So we have $4! \cdot 6!$ ways of arranging the 9 cars such that the four passenger cars are together. There is a total of 9! ways of arranging all 9 individual cars, so we have

$$P(\text{gentleman must pass only through passenger cars}) = \frac{6!4!}{9!} = \frac{1}{21}.$$

(c) There are 14 head of cattle to be transported in the three cattle cars.

i. In how many ways can 5, 5, and 4 head of cattle, respectively, be put into the three cattle cars?

This is a partitioning of the 14 head of cattle into groups of 5, 5, and 4. The number of ways to do this is

$$\frac{14!}{5!5!4!} = 252252.$$

ii. The gallant gentleman owns 3 of the 14 head of cattle. If 5, 5, and 4 of the 14 head of cattle are put into the three cattle cars at random, with what probability will the gallant gentleman's cattle all be placed in the same cattle car?

The number of ways in which his 3 head of cattle can be put together is given by

$$\frac{11!}{2!5!4!} + \frac{11!}{5!2!4!} + \frac{11!}{5!5!1!} = 6930 + 6930 + 2772 = 16632.$$

which we get by considering placing them in the first, the second, or the third cattle car. Since there are 252252 ways to partition the 14 head of cattle into groups of sizes 5, 5, and 4, the probability that the gallant gentleman's cattle are placed together is

$$\frac{16632}{252252} = 0.06593407.$$

(d) Mr. and Mrs. Wilkins and their two daughters and three sons are boarding one of the passenger cars on the next stop.

i. In how many different orders can the members of the Wilkins family enter the car?

The family members can enter the car in $7! = 5040$ orders.

ii. In how many of these orders does Mrs. Wilkins precede Mr. Wilkins?

Mrs. Wilkins precedes Mr. Wilkins in one half of the orders, that is in 2520 orders.

iii. In how many of these orders does Mrs. Wilkins and her two daughters precede Mr. Wilkins his three sons?

> Mrs. Wilkins and her two daughters precede Mr. Wilkins and his three sons in $3!4! = 144$ of the orders.

(e) At each of the 10 stops along Lil' Jonnie's journey on the train, he will decide whether to quickly hop out and scrawl his initials somewhere around the station or to remain on board. At how many unique sets of stations can Lil' Jonnie scrawl his initials?

> Lil' Johnnie performs a sequence of 10 tasks, each of which can be done in 2 ways, so the job can be done in $2^{10} = 1024$ ways.

4. Suppose there are 5 bowling balls which are identical except that one is magical and delivers, no matter what, a strike with probability 3/4. Suppose you get a strike 1 out of 4 times on average when using non-magical bowling balls. You select one of the 5 balls at random and send it down the lane...

(a) Give the probability that you get a strike.

> Let S be the event of a strike and let M be the event of choosing the magic ball. The problem gives $P(S|M) = 3/4$, $P(S|M^c) = 1/4$ and $P(M) = 1/5$. We have
>
> $$P(S) = P(S|M)P(M) + P(S|M^c)P(M^c) = \frac{3}{4} \cdot \frac{1}{5} + \frac{1}{4} \cdot \frac{4}{5} = \frac{7}{20}$$

(b) Given that you got a strike, what is the probability you chose the magic bowling ball?

> We have
>
> $$P(M|S) = \frac{P(S|M)P(M)}{P(S|M)P(M) + P(S|M^c)P(M^c)} = \frac{3/20}{7/20} = \frac{3}{7}.$$

(c) Suppose you choose a ball and with the same ball you get two strikes in a row. What is the probability that you chose the magic ball?

> Let S_1 and S_2 be the event of getting a strike the first and the second time you send the ball down the lane, respectively. We have $P(S_1 \cap S_2|M) = P(S_1|M)P(S_2|M) = 9/16$, assuming independence of S_1 and S_2 given M. Also, $P(S_1 \cap S_2|M^c) = P(S_1|M^c)P(S_2|M^c) = 1/16$. So we have
>
> $$P(M|S_1 \cap S_2) = \frac{P(S_1 \cap S_2|M)P(M)}{P(S_1 \cap S_2|M)P(M) + P(S_1 \cap S_2|M)^c P(M^c)} = \frac{9/16}{9/16 + 1/16} = \frac{9}{10}.$$

5. Consider a bag of marbles, 19 of which are green, 25 of which are blue, and 6 of which are red. Moreover, suppose 9 of the green marbles are opaque, 5 of the blue marbles are opaque, and 3 of the red marbles are opaque, and the rest of the marbles are transparent. Suppose you draw one marble from the bag and let G, B, and R be the events that the marble is green, blue, and red, respectively, and let O be the event that it is opaque.

It helps to make a table:

	Green	Blue	Red	Total
Opaque	9	5	3	17
Transparent	10	20	3	33
Total	19	25	6	50

(a) Find $P(R|O)$.

$P(R|O) = 3/17$.

(b) Find $P(R|O^c)$.

$P(R|O^c) = 3/33$.

(c) Find $P(R)$.

$P(R) = 6/50$.

(d) Check whether R and O are independent.

We have $P(R) = 6/50 \neq P(R|O) = 3/17$, so R and O are not independent.

(e) Find $P(B^c|O)$.

$P(B^c|O) = 12/17$.

(f) Find $P(O|G)$.

$P(O|G) = 9/19$.

(g) Find $P(B \cup G|O)$.

$P(B \cup G|O) = 14/17$.

6. In all the state of the union (SOTU) addresses since that of President George Washington in the year 1790 to that of President Barack Obama in the year 2016, a total of 69,158 sentences were spoken. In all SOTU addresses following that of President Woodrow Wilson in 1920, a total of 32,752 sentences were spoken; among these, 15,934 had a length or 20 words or more. Among the sentences spoken prior to this, 27,639 had a length of 20 words or more.
See `https://doi.org/10.46430/phen0061` *if you want to play with this kind of data.*

   (a) If you select a sentence at random from the SOTU addresses given after 1920, with what probability does it have a length of at least 20 words?

   > Let L be the event that the sentence has a length greater than 20 words and A be the event that the sentence was delivered after 1920. Then we have
   > $$P(L|A) = \frac{15934}{32752} = 0.4865046.$$

   (b) If you select a sentence at random from the SOTU addresses given before or during 1920, with what probability does it have a length of at least 20 words?

   > We have
   > $$P(L|A^c) = \frac{27639}{69158 - 32752} = \frac{27639}{36406} = 0.759188.$$

   (c) If you select a sentence at random from all the SOTU addresses from 1790 to 2016, with what probability does it have a length of at least 20 words?

   > We have
   > $$P(L) = \frac{15934 + 27639}{69158} = 0.63005.$$

   (d) If you select a sentence at random from all the SOTU addresses from 1790 to 2016, what is the probability that it belongs to a SOTU address given after 1920, given that it has a length of at least 20 words?

   > We have
   > $$P(A|L) = \frac{P(A \cap L)}{P(L)} = \frac{P(L|A)P(A)}{P(L)} = \frac{0.4865046 \cdot 0.4735822}{0.63005} = 0.365685,$$
   > where $P(A) = 0.4735822 = 32752/69158$.

7. Suppose you order a new pair of swimming goggles and that they will be manufactured with a defect making them leaky with probability $1/40$.

   (a) What is the probability of receiving a defective pair, and then again receiving a defective pair when you re-order the goggles?

Let D_1 and D_2 be the event that the goggles are defective the first and second time ordered, respectively. Then we have

$$P(D_1 \cap D_2) = P(D_1)P(D_2) = (1/40)^2 = 0.000625.$$

(b) What assumption did you make in order to compute your answer to part (a)?

The assumption was that D_1 and D_2 were independent events.

(c) What is the probability that you receive a defective pair and then a functioning pair when you re-order the goggles?

We have

$$P(D_1 \cap D_2^c) = P(D_1)P(D_2^c) = (1/40)(39/40) = 39/1600 = 0.024375.$$

(d) Give the probability in terms of K that you receive K defective pairs of goggles, where $K \geq 1$.

Letting D_i be the event that the goggles are defective the ith time ordered, for $i = 1, \ldots, K$, we have

$$P(D_1 \cap \cdots \cap D_K) = P(D_1) \times \cdots \times P(D_K) = (1/40)^K,$$

assuming that $D_1, \ldots, D_K$ are mutually independent.